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Signals and Systems Formula Sheet

INSIDE

- 249 formulas in 9 topics
- When to use each one
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EVEN AND ODD PARTS

$$x_e(t) = \frac{x(t) + x(-t)}{2}, \qquad x_o(t) = \frac{x(t) - x(-t)}{2}, \qquad x(t) = x_e(t) + x_o(t)$$

Every signal splits this way, uniquely; the DT form is identical with n in place of t. Example: $u(t) = \frac{1}{2} + \frac{1}{2}\,\mathrm{sgn}(t)$, so $u_e(t) = \frac{1}{2}$ and $u_o(t) = \frac{1}{2}\,\mathrm{sgn}(t)$ for $t \neq 0$.

UNIT STEP, SIGNUM AND RAMP

$$u(t) = \begin{cases} 1, & t > 0 \\ 0, & t < 0 \end{cases}, \qquad \mathrm{sgn}(t) = 2u(t) - 1, \qquad r(t) = t\,u(t) = \int_{-\infty}^{t} u(\tau)\,d\tau$$

$\mathrm{sgn}(0) = 0$, so $\mathrm{sgn}(t) = 2u(t) - 1$ holds for $t \neq 0$. Derivative chain: $\frac{d}{dt} r(t) = u(t)$, $\frac{d}{dt} u(t) = \delta(t)$, $\frac{d}{dt}\,\mathrm{sgn}(t) = 2\delta(t)$.

RECTANGULAR AND TRIANGULAR PULSES

$$\mathrm{rect}\left(\frac{t}{\tau}\right) = \begin{cases} 1, & |t| < \tau/2 \\ 0, & |t| > \tau/2 \end{cases}, \qquad \mathrm{tri}\left(\frac{t}{\tau}\right) = \begin{cases} 1 - |t|/\tau, & |t| \leq \tau \\ 0, & |t| > \tau \end{cases}$$

For $\tau > 0$: rect has width τ and area τ; tri has half-width τ (base 2τ), peak 1 and area τ. Also $\mathrm{rect}(t) * \mathrm{rect}(t) = \mathrm{tri}(t)$. Some texts write the triangle with total base τ; read the definition given.

SIFTING AND SAMPLING PROPERTIES OF THE IMPULSE

$$\int_{t_1}^{t_2} x(t)\,\delta(t - t_0)\,dt = \begin{cases} x(t_0), & t_1 < t_0 < t_2 \\ 0, & t_0 \notin [t_1, t_2] \end{cases}, \qquad x(t)\,\delta(t - t_0) = x(t_0)\,\delta(t - t_0)$$

Requires x continuous at t_0. When t_0 sits exactly on a limit, the value depends on a convention, so that case is left out. The impulse has unit area, $\int_{-\infty}^{\infty} \delta(t)\,dt = 1$, and $u(t) = \int_{-\infty}^{t} \delta(\tau)\,d\tau$.

TIME SCALING OF THE IMPULSE

$$\delta(at + b) = \frac{1}{|a|}\,\delta\left(t + \frac{b}{a}\right), \qquad a \neq 0$$

With $a = -1$: $\delta(-t) = \delta(t)$, so the impulse is even. Hence $\int_{-\infty}^{\infty} x(t)\,\delta(at + b)\,dt = \frac{1}{|a|}\,x\left(-\frac{b}{a}\right)$.

IMPULSE OF A FUNCTION

$$\delta\big(f(t)\big) = \sum_{k} \frac{\delta(t - t_k)}{|f'(t_k)|}$$

The sum runs over the roots t_k of $f(t) = 0$, and every root must be simple ($f'(t_k) \neq 0$); at a repeated root $\delta(f(t))$ is not defined. Example: $\delta(t^2 - 9) = \frac{1}{6}\,[\delta(t - 3) + \delta(t + 3)]$.

DERIVATIVES OF THE IMPULSE

$$\int_{-\infty}^{\infty} x(t)\,\delta^{(n)}(t - t_0)\,dt = (-1)^n\,x^{(n)}(t_0)$$

Requires $x^{(n)}$ continuous at t_0. For $n = 1$ (the doublet): $\int_{-\infty}^{\infty} x(t)\,\delta'(t - t_0)\,dt = -x'(t_0)$. The doublet $\delta'(t)$ is odd, and $x(t)\,\delta'(t) = x(0)\,\delta'(t) - x'(0)\,\delta(t)$, not just $x(0)\,\delta'(t)$.

DISCRETE-TIME IMPULSE AND STEP

$$\delta[n] = u[n] - u[n-1], \qquad u[n] = \sum_{k=0}^{\infty} \delta[n-k] = \sum_{m=-\infty}^{n} \delta[m], \qquad \sum_{n=-\infty}^{\infty} x[n]\,\delta[n - n_0] = x[n_0]$$

$\delta[0] = 1$ is an ordinary finite value, unlike $\delta(t)$. $\delta[an] = \delta[n]$ for every nonzero integer a: there is no $1/|a|$ factor in DT.

ANY SIGNAL AS A WEIGHTED SUM OF SHIFTED IMPULSES

$$x[n] = \sum_{k=-\infty}^{\infty} x[k]\,\delta[n-k], \qquad x(t) = \int_{-\infty}^{\infty} x(\tau)\,\delta(t-\tau)\,d\tau$$

The DT form holds sample by sample, since only the $k = n$ term survives. The CT form is the sifting property with $\delta(t-\tau) = \delta(\tau - t)$, valid wherever x is continuous. Both are the starting point for convolution.

WHERE A POINT OF X LANDS UNDER X(AT+B)

$$y(t) = x(at+b),\ a \neq 0: \quad \text{the value } x(t_1) \text{ appears at } t = \frac{t_1 - b}{a}$$

Map each breakpoint of x this way. A support $[t_1, t_2]$ becomes $\left[\frac{t_1-b}{a}, \frac{t_2-b}{a}\right]$, with the ends swapped when $a < 0$. This check does not depend on any order of operations.

EFFECT OF TIME OPERATIONS ON AREA, ENERGY AND POWER

$$\int_{-\infty}^{\infty} x(at+b)\,dt = \frac{1}{|a|}\int_{-\infty}^{\infty} x(t)\,dt, \qquad E_{x(at+b)} = \frac{E_x}{|a|}, \qquad P_{x(at+b)} = P_x \ \ (x \text{ periodic}), \qquad a \neq 0$$

Shifts change none of these. Time scaling divides area and energy by $|a|$ but leaves the power, average and RMS value of a periodic signal unchanged (its period becomes $T_0/|a|$). Amplitude scaling by k multiplies E and P by $|k|^2$.

CT SINUSOID AND SUMS OF CT PERIODIC SIGNALS

$$T = \frac{2\pi}{\omega_0}\ (\omega_0 > 0); \qquad \frac{T_1}{T_2} = \frac{p}{q} \text{ rational } \Rightarrow\ x_1 + x_2 \text{ periodic, with period } \mathrm{LCM}(T_1, T_2) = qT_1 = pT_2$$

Take p/q in lowest terms. For non-constant x_1, x_2 with fundamental periods T_1, T_2 the converse also holds: if T_1/T_2 is irrational, $x_1 + x_2$ is not periodic. The LCM is always a period of the sum, and it is the fundamental period unless components cancel: $x_1 = \cos t + \cos 2t$ and $x_2 = -\cos t$ both have period 2π, but their sum $\cos 2t$ has period π. For a sum of sinusoids with nonzero amplitudes at distinct positive frequencies, the fundamental angular frequency is the GCD of the component frequencies, $\omega_0 = \gcd(\omega_1, \omega_2, \dots)$, and $T_0 = 2\pi/\omega_0$; a DC term does not change this.

DT SINUSOID PERIODICITY

$$\cos(\Omega_0 n + \theta) \text{ periodic } \iff \frac{\Omega_0}{2\pi} = \frac{m}{N} \text{ rational}; \qquad N_0 = N \ \text{ with } \frac{m}{N} \text{ in lowest terms, } N > 0$$

Ω_0 is in rad/sample. Equivalently, N_0 is the smallest positive integer N for which $\Omega_0 N$ is a multiple of 2π; for $\Omega_0 \neq 0$ that is $N_0 = 2\pi m/\Omega_0$ with the smallest positive integer m that makes it an integer. The same test applies to $e^{j\Omega_0 n}$. The one exception is a sequence that is identically zero, such as $\sin(\pi n)$, which trivially has period 1. A sum of periodic sequences is always periodic: $\mathrm{LCM}(N_1, N_2)$ is a period, and it is the fundamental period unless components cancel.

DT FREQUENCY IS DEFINED ONLY MODULO 2Π

$$e^{j(\Omega_0 + 2\pi k)n} = e^{j\Omega_0 n}, \qquad \cos\big((2\pi - \Omega_0)n\big) = \cos(\Omega_0 n)$$

For integers k and n. Distinct DT sinusoids need only $\Omega \in (-\pi, \pi]$ (or $[0, \pi]$ for real cosines). The fastest oscillation is at $\Omega = \pi$: $\cos(\pi n) = (-1)^n$.

ENERGY

$$E = \int_{-\infty}^{\infty} |x(t)|^2\,dt, \qquad E = \sum_{n=-\infty}^{\infty} |x[n]|^2$$

Normalised energy, as if x were a voltage across $1\ \Omega$. Use $|x|^2$, not x^2, for complex signals.

AVERAGE POWER

$$P = \lim_{T\to\infty} \frac{1}{2T}\int_{-T}^{T} |x(t)|^2\,dt, \qquad P = \lim_{N\to\infty} \frac{1}{2N+1}\sum_{n=-N}^{N} |x[n]|^2$$

The window is symmetric about zero, which is why $A\,u(t)$ has power $A^2/2$. If $E < \infty$ then $P = 0$; if $0 < P < \infty$ then $E = \infty$.

POWER OF A PERIODIC SIGNAL

$$P = \frac{1}{T_0}\int_{t_0}^{t_0+T_0} |x(t)|^2\,dt, \qquad P = \frac{1}{N_0}\sum_{n=n_0}^{n_0+N_0-1} |x[n]|^2$$

Any one full period, from any starting point. For a periodic signal, $P = X_{\text{rms}}^2$.

POWER OF DC PLUS SINUSOIDS

$$x(t) = A_0 + \sum_k A_k \cos(\omega_k t + \theta_k) \;\; (\omega_k \text{ distinct and positive}) \;\Rightarrow\; P = A_0^2 + \sum_k \frac{A_k^2}{2}$$

For a finite sum with real amplitudes, whether or not x is periodic. Terms at the same frequency must first be merged into one sinusoid; a negative frequency is the same frequency, since $\cos(-\omega t + \theta) = \cos(\omega t - \theta)$. For $\sum_k C_k e^{j\omega_k t}$ with distinct ω_k (here ω and $-\omega$ do count as different), $P = \sum_k |C_k|^2$.

MERGING TWO SINUSOIDS OF THE SAME FREQUENCY

$$A_1\cos(\omega t + \theta_1) + A_2\cos(\omega t + \theta_2) = A\cos(\omega t + \phi), \qquad A^2 = A_1^2 + A_2^2 + 2A_1A_2\cos(\theta_1 - \theta_2)$$

This is phasor addition, $A\angle\phi = A_1\angle\theta_1 + A_2\angle\theta_2$. The powers add directly only when $\cos(\theta_1 - \theta_2) = 0$; for example $a\cos\omega t + b\sin\omega t$ has amplitude $\sqrt{a^2+b^2}$.

AVERAGE (DC) VALUE AND RMS VALUE

$$V_{\text{avg}} = \frac{1}{T}\int_{t_0}^{t_0+T} v(t)\,dt, \qquad V_{\text{rms}} = \sqrt{\frac{1}{T}\int_{t_0}^{t_0+T} v^2(t)\,dt}$$

Over any one full period of a real waveform $v(t)$. V_{rms}^2 is the average power in $1\ \Omega$: the RMS value is the DC level that would dissipate the same power in a resistor.

DC AND AC PARTS

$$V_{\text{rms}}^2 = V_{\text{avg}}^2 + V_{\text{ac,rms}}^2$$

The AC part $v(t) - V_{\text{avg}}$ has zero mean, so it is orthogonal to the DC part. For $v = V_0 + \sum_k V_k\cos(\omega_k t + \theta_k)$ with distinct positive ω_k: $V_{\text{rms}} = \sqrt{V_0^2 + \sum_k V_k^2/2}$. For a waveform with nonzero mean, the ratio $V_{\text{ac,rms}}/V_{\text{avg}} = \sqrt{\text{FF}^2 - 1}$ (FF as defined below, with the full-period average) is the ripple factor used for rectifier outputs.

STRAIGHT-LINE SEGMENT SHORTCUT

$$\text{segment from } a \text{ to } b: \qquad \text{mean} = \frac{a+b}{2}, \qquad \text{mean square} = \frac{a^2+ab+b^2}{3}$$

Valid for any straight-line piece, whatever its slope or duration. Weight each segment's mean (or mean square) by its fraction of the period and add. A flat piece at level c is the case $a = b = c$.

FORM FACTOR AND CREST (PEAK) FACTOR

$$\text{FF} = \frac{V_{\text{rms}}}{V_{\text{avg}}}, \qquad \text{CF} = \frac{V_{\text{peak}}}{V_{\text{rms}}}$$

For a waveform with nonzero mean, V_{avg} is the full-period average. For a zero-mean AC wave that average is zero, so V_{avg} in FF means the average of $|v(t)|$ over a period, which for a sine, square or symmetric triangle wave equals the average over a positive half-cycle. Sine: FF $= \frac{\pi}{2\sqrt{2}} \approx 1.11$, CF $= \sqrt{2} \approx 1.414$. Square: FF $=$ CF $= 1$. Triangle: FF $= \frac{2}{\sqrt{3}} \approx 1.155$, CF $= \sqrt{3} \approx 1.732$. Half-wave rectified sine: FF $= \frac{\pi}{2} \approx 1.571$, CF $= 2$. Full-wave rectified sine: the same as the sine, FF ≈ 1.11, CF $= \sqrt{2}$.

LTI systems: properties, impulse response, convolution and correlation

LINEARITY (SUPERPOSITION)

$$T\{a\,x_1 + b\,x_2\} = a\,T\{x_1\} + b\,T\{x_2\}$$

Must hold for all inputs x_1, x_2 and all constants a, b (complex constants if the signals may be complex). It combines additivity ($a = b = 1$) and homogeneity ($b = 0$). Zero input giving zero output is necessary, not sufficient.

TIME INVARIANCE

$$x(t) \to y(t) \;\Rightarrow\; x(t - t_0) \to y(t - t_0) \quad \text{for every } t_0$$

Discrete time: $x[n - n_0] \to y[n - n_0]$ for every integer n_0. Also called shift invariance.

CAUSALITY AND MEMORY

$$\text{causal: } y(t_0) \text{ depends only on } x(t),\ t \le t_0 \qquad \text{memoryless: } y(t_0) \text{ depends only on } x(t_0)$$

Both must hold for every t_0. A memoryless (static) system is always causal. A causal system need not be memoryless.

BIBO STABILITY

$$|x(t)| \le M_x < \infty \text{ for all } t \;\Rightarrow\; |y(t)| \le M_y < \infty \text{ for all } t$$

Every bounded input must give a bounded output. One bounded input with an unbounded output proves the system unstable.

CONVOLUTION INTEGRAL

$$y(t) = x(t) * h(t) = \int_{-\infty}^{\infty} x(\tau)\,h(t - \tau)\,d\tau = \int_{-\infty}^{\infty} h(\tau)\,x(t - \tau)\,d\tau$$

τ is the integration variable and t is held fixed while integrating. In practice the limits shrink to the interval where both factors are nonzero.

CONVOLUTION SUM

$$y[n] = x[n] * h[n] = \sum_{k=-\infty}^{\infty} x[k]\,h[n - k] = \sum_{k=-\infty}^{\infty} h[k]\,x[n - k]$$

Each output sample is a sum of products of samples whose indices add to n.

CAUSAL SYSTEM WITH A CAUSAL INPUT

$$y(t) = \int_{0}^{t} x(\tau)\,h(t - \tau)\,d\tau \;\; (t \ge 0), \qquad y[n] = \sum_{k=0}^{n} x[k]\,h[n - k] \;\; (n \ge 0)$$

When x and h are both zero before the origin, y is zero there too. If either signal has an impulse at the origin, take the lower limit as 0^- so that the impulse is included.

CAUSALITY OF AN LTI SYSTEM

$$h(t) = 0 \text{ for } t < 0, \qquad h[n] = 0 \text{ for } n < 0$$

Then $y(t) = \int_0^{\infty} h(\tau)\,x(t - \tau)\,d\tau$ uses only present and past input. Necessary and sufficient for an LTI system.

BIBO STABILITY OF AN LTI SYSTEM

$$\int_{-\infty}^{\infty} |h(t)|\,dt < \infty, \qquad \sum_{n=-\infty}^{\infty} |h[n]| < \infty$$

Necessary and sufficient for an LTI system. The integral of $|h|$ must be finite, not the integral of h. The output is then bounded by $|y| \le M_x \int |h|\,dt$. If $h(t)$ contains impulses, each term $c\,\delta(t - t_i)$ adds $|c|$ to the integral, so $h(t) = \delta(t)$ (a plain wire) is stable. A doublet term $\delta'(t)$ (a differentiator) makes the system unstable.

STEP RESPONSE (CONTINUOUS TIME)

$$s(t) = h(t) * u(t) = \int_{-\infty}^{t} h(\tau)\, d\tau, \qquad h(t) = \frac{ds(t)}{dt}$$

Here $s(t)$ is the step response, not the Laplace variable. The step response is the running integral of h. For a causal h, $s(t) = \int_0^t h(\tau)\, d\tau$. For a stable system the final value is $s(\infty) = \int_{-\infty}^{\infty} h(t)\, dt = H(0)$, the DC gain. The ramp response is the running integral of the step response.

STEP RESPONSE (DISCRETE TIME)

$$s[n] = h[n] * u[n] = \sum_{k=-\infty}^{n} h[k], \qquad h[n] = s[n] - s[n-1]$$

The step response is the running sum of h, and h is the first difference of the step response.

INTERCONNECTIONS OF LTI SYSTEMS

$$\text{cascade: } h = h_1 * h_2 = h_2 * h_1, \qquad \text{parallel: } h = h_1 + h_2$$

In any transform domain a cascade multiplies, $H = H_1 H_2$, and a parallel connection adds, $H = H_1 + H_2$. The order of LTI blocks in a cascade does not change the overall response.

INVERSE SYSTEM

$$h(t) * h_i(t) = \delta(t), \qquad H(s)\, H_i(s) = 1$$

Examples: the delay $\delta(t - t_0)$ has inverse $\delta(t + t_0)$. The accumulator $h[n] = u[n]$ has inverse $\delta[n] - \delta[n-1]$, because $u[n] - u[n-1] = \delta[n]$. An inverse need not be causal or stable: for $t_0 > 0$ the delay is causal, but its inverse, an advance, is not.

LTI SYSTEM FROM A LINEAR CONSTANT-COEFFICIENT EQUATION

$$\sum_{k=0}^{N} a_k \frac{d^k y(t)}{dt^k} = \sum_{k=0}^{M} b_k \frac{d^k x(t)}{dt^k} \;\Rightarrow\; H(s) = \frac{\sum_{k=0}^{M} b_k\, s^k}{\sum_{k=0}^{N} a_k\, s^k}$$

$$\sum_{k=0}^{N} a_k\, y[n-k] = \sum_{k=0}^{M} b_k\, x[n-k] \;\Rightarrow\; H(z) = \frac{\sum_{k=0}^{M} b_k\, z^{-k}}{\sum_{k=0}^{N} a_k\, z^{-k}}$$

Holds when the system is at rest before the input starts (initial rest), which makes it LTI and causal. h is the inverse transform with the ROC to the right of the rightmost pole of $H(s)$, or outside the outermost pole of $H(z)$. Examples: $\frac{dy}{dt} + a\, y = x(t)$ gives $H(s) = \frac{1}{s + a}$ and $h(t) = e^{-at} u(t)$, stable only for $a > 0$. $y[n] - a\, y[n-1] = x[n]$ gives $H(z) = \frac{1}{1 - a z^{-1}}$ and $h[n] = a^n u[n]$, stable only for $|a| < 1$. An $h[n]$ of finite length is called FIR and one of infinite length IIR. A difference equation with no past outputs ($N = 0$) is always FIR, with $h[n] = b_n / a_0$.

COMMUTATIVE, ASSOCIATIVE AND DISTRIBUTIVE LAWS

$$x * h = h * x, \qquad (x * h_1) * h_2 = x * (h_1 * h_2), \qquad x * (h_1 + h_2) = x * h_1 + x * h_2$$

Hold for continuous and discrete convolution. Associativity gives the cascade rule and distributivity gives the parallel rule.

CONVOLUTION WITH AN IMPULSE

$$x(t) * \delta(t) = x(t), \qquad x(t) * \delta(t - t_0) = x(t - t_0), \qquad \delta(t - t_1) * \delta(t - t_2) = \delta(t - t_1 - t_2)$$

Discrete time: $x[n] * \delta[n - n_0] = x[n - n_0]$. Convolving with an impulse train repeats the signal: $x(t) * \sum_k \delta(t - kT_0) = \sum_k x(t - kT_0)$. Do not confuse convolution with multiplication by an impulse, which samples: $x(t)\, \delta(t - t_0) = x(t_0)\, \delta(t - t_0)$, for x continuous at t_0.

SHIFT PROPERTY

$$x(t) * h(t) = y(t) \;\Rightarrow\; x(t - t_1) * h(t - t_2) = y(t - t_1 - t_2)$$

Shifts add. The same holds in discrete time with integer shifts.

CONVOLUTION WITH THE UNIT STEP

$$x(t) * u(t) = \int_{-\infty}^{t} x(\tau)\, d\tau, \qquad x[n] * u[n] = \sum_{k=-\infty}^{n} x[k]$$

Convolving with u is running integration (or running summation). Convolving with $\delta'(t)$ differentiates: $x(t) * \delta'(t) = x'(t)$.

DIFFERENTIATION PROPERTY

$$\frac{d}{dt}\left[x(t) * h(t)\right] = \frac{dx}{dt} * h = x * \frac{dh}{dt}$$

Useful for piecewise-constant signals, whose derivatives are impulses. A result found this way must be integrated back, using the fact that y is zero before $t = a_1 + a_2$.

AREA (SUM) PROPERTY

$$\int_{-\infty}^{\infty} y(t)\, dt = \left(\int_{-\infty}^{\infty} x(t)\, dt\right)\left(\int_{-\infty}^{\infty} h(t)\, dt\right), \qquad \sum_{n} y[n] = \left(\sum_{n} x[n]\right)\left(\sum_{n} h[n]\right)$$

Holds whenever both areas (sums) are finite, for example when x and h are absolutely integrable (summable). The quickest check on any such convolution. In transform terms it is $Y(0) = X(0)\, H(0)$.

WIDTH AND LENGTH RULE

$$x \text{ on } [a_1, b_1],\ h \text{ on } [a_2, b_2] \ \Rightarrow\ y \text{ on } [a_1 + a_2,\ b_1 + b_2]$$

$$\text{lengths } N_1, N_2 \ \Rightarrow\ N_1 + N_2 - 1 \text{ samples, from } n_1 + n_2 \text{ to } n_1 + n_2 + N_1 + N_2 - 2$$

Continuous time: the widths add, $T_y = T_x + T_h$. Discrete time: n_1 and n_2 are the first indices of x and h, and each length is counted from the first nonzero sample to the last. The first and last output samples are then single products of nonzero end samples, so they are nonzero; interior samples may be zero.

TIME-SCALING PROPERTY

$$x(at) * h(at) = \frac{1}{|a|}\, y(at), \qquad a \neq 0$$

Continuous time only. Scaling both signals by a compresses the output time axis by a and scales its height by $1/|a|$.

CONVOLUTION THEOREM

$$y = x * h \ \leftrightarrow\ Y(j\omega) = X(j\omega)\, H(j\omega), \quad Y(s) = X(s)\, H(s), \quad Y(z) = X(z)\, H(z)$$

The ROC of Y contains the intersection of the ROCs of X and H. Multiplication in time is convolution in frequency: $x(t)\, p(t) \leftrightarrow \frac{1}{2\pi} X(j\omega) * P(j\omega)$ in the ω convention, and $X(f) * P(f)$, without the $1/2\pi$, in the f convention.

EIGENFUNCTION PROPERTY (CONTINUOUS TIME)

$$x(t) = e^{s_0 t} \ \Rightarrow\ y(t) = H(s_0)\, e^{s_0 t}, \qquad H(s) = \int_{-\infty}^{\infty} h(\tau)\, e^{-s\tau}\, d\tau$$

Holds for all t when s_0 lies in the ROC of $H(s)$, the bilateral Laplace transform of h. The input exists for all time, $-\infty < t < \infty$.

EIGENFUNCTION PROPERTY (DISCRETE TIME)

$$x[n] = z_0^n \ \Rightarrow\ y[n] = H(z_0)\, z_0^n, \qquad H(z) = \sum_{k=-\infty}^{\infty} h[k]\, z^{-k}$$

z_0 must lie in the ROC of $H(z)$. With $z_0 = e^{j\Omega_0}$ the eigenvalue is the frequency response $H(e^{j\Omega_0})$.

FREQUENCY RESPONSE

$$H(j\omega) = \int_{-\infty}^{\infty} h(t)\, e^{-j\omega t}\, dt = |H(j\omega)|\, e^{j\angle H(j\omega)}, \qquad H(e^{j\Omega}) = \sum_{n=-\infty}^{\infty} h[n]\, e^{-j\Omega n}$$

Exists as an ordinary function when h is absolutely integrable (summable), that is, when the system is BIBO stable. For a real h, $|H|$ is even and $\angle H$ is odd in frequency.

LTI RESPONSE TO A SINUSOID

$$A\cos(\omega_0 t + \phi) \;\rightarrow\; A\,|H(j\omega_0)|\cos\left(\omega_0 t + \phi + \angle H(j\omega_0)\right)$$

Needs a real h and a stable system. Discrete time: $A\cos(\Omega_0 n + \phi) \rightarrow A\,|H(e^{j\Omega_0})|\cos(\Omega_0 n + \phi + \angle H(e^{j\Omega_0}))$. For a sum of sinusoids, apply the rule to each term and add (superposition).

RESPONSE TO A CONSTANT (DC GAIN)

$$x(t) = K \;\Rightarrow\; y(t) = K\,H(0) = K\int_{-\infty}^{\infty} h(t)\, dt, \qquad x[n] = K \;\Rightarrow\; y[n] = K\sum_{k} h[k]$$

The $\omega = 0$ (or $\Omega = 0$) case of the sinusoid rule. For $x[n] = (-1)^n$ use $\Omega = \pi$: $y[n] = H(e^{j\pi})\,(-1)^n$, with $H(e^{j\pi}) = \sum_k (-1)^k h[k]$.

CROSS-CORRELATION (ENERGY SIGNALS)

$$R_{xy}(\tau) = \int_{-\infty}^{\infty} x(t+\tau)\, y^*(t)\, dt = \int_{-\infty}^{\infty} x(t)\, y^*(t-\tau)\, dt = x(\tau) * y^*(-\tau)$$

Correlation is convolution with the second signal reversed and conjugated. Because of the reversal it is **not** commutative: $R_{yx}(\tau) = R_{xy}^*(-\tau)$.

AUTOCORRELATION (ENERGY SIGNALS)

$$R_{xx}(\tau) = \int_{-\infty}^{\infty} x(t+\tau)\, x^*(t)\, dt = x(\tau) * x^*(-\tau)$$

For a real, even signal, $x(-\tau) = x(\tau)$, so $R_{xx} = x * x$.

DISCRETE-TIME CORRELATION

$$R_{xy}[m] = \sum_{n=-\infty}^{\infty} x[n+m]\, y^*[n] = x[m] * y^*[-m], \qquad R_{xx}[m] = \sum_{n=-\infty}^{\infty} x[n+m]\, x^*[n]$$

m is the integer lag. The length rule of convolution applies: sequences of lengths N_1 and N_2 give $N_1 + N_2 - 1$ lags, so the autocorrelation of an N-sample sequence has $2N - 1$ lags.

PROPERTIES OF AUTOCORRELATION

$$R_{xx}(0) = \int_{-\infty}^{\infty} |x(t)|^2\, dt = E_x$$
$$R_{xx}(-\tau) = R_{xx}^*(\tau)$$
$$|R_{xx}(\tau)| \le R_{xx}(0)$$
$$\int_{-\infty}^{\infty} R_{xx}(\tau)\, d\tau = \left|\int_{-\infty}^{\infty} x(t)\, dt\right|^2$$

For a real signal, R_{xx} is real and even, and its largest value is at $\tau = 0$. The last line is the area property of convolution; it needs x absolutely integrable.

PROPERTIES OF CROSS-CORRELATION

$$R_{xy}(\tau) = R_{yx}^*(-\tau)$$
$$|R_{xy}(\tau)|^2 \le E_x\, E_y$$

The inequality is the Cauchy-Schwarz inequality. Unlike R_{xx}, the cross-correlation need not peak at $\tau = 0$.

CORRELATION AND ENERGY SPECTRAL DENSITY

$$R_{xx}(\tau) \leftrightarrow |X(j\omega)|^2, \qquad R_{xy}(\tau) \leftrightarrow X(j\omega)\,Y^*(j\omega), \qquad E_x = \frac{1}{2\pi}\int_{-\infty}^{\infty} |X(j\omega)|^2\,d\omega$$

$|X(j\omega)|^2$ is the energy spectral density. In the f convention, $E_x = \int |X(f)|^2\,df$ (Parseval). Discrete time: $R_{xx}[m] \leftrightarrow |X(e^{j\Omega})|^2$ and $E_x = \sum_n |x[n]|^2 = \frac{1}{2\pi}\int_{-\pi}^{\pi} |X(e^{j\Omega})|^2\,d\Omega$. Because $|X|^2$ discards phase, $x(t)$ and every shifted copy $x(t-t_0)$ have the same autocorrelation.

AUTOCORRELATION OF POWER SIGNALS

$$R_{xx}(\tau) = \lim_{T\to\infty}\frac{1}{T}\int_{-T/2}^{T/2} x(t+\tau)\,x^*(t)\,dt, \qquad R_{xx}(0) = P_x$$

For a periodic signal of period T_0, average over one period; R_{xx} is then periodic with period T_0 and returns to its peak $R_{xx}(0)$ at every multiple of T_0. The cross-correlation of power signals uses the same average with $y^*(t)$ in place of $x^*(t)$. The power spectral density is the Fourier transform of the autocorrelation, $S_{xx}(\omega) = \mathcal{F}\{R_{xx}(\tau)\}$, and $P_x = \frac{1}{2\pi}\int_{-\infty}^{\infty} S_{xx}(\omega)\,d\omega$. Example: $A\cos(\omega_0 t + \phi)$ has $R_{xx}(\tau) = \frac{A^2}{2}\cos\omega_0\tau$, whatever ϕ is, so $R_{xx}(0) = A^2/2$ is its average power and $S_{xx}(\omega) = \frac{\pi A^2}{2}[\delta(\omega-\omega_0) + \delta(\omega+\omega_0)]$.

CORRELATION THROUGH AN LTI SYSTEM

$$R_{yx}(\tau) = h(\tau) * R_{xx}(\tau), \qquad R_{yy}(\tau) = R_{hh}(\tau) * R_{xx}(\tau), \qquad R_{hh}(\tau) = h(\tau) * h^*(-\tau)$$

For $y = h * x$, in the convention above. In frequency, $S_{yx}(\omega) = H(j\omega)\,S_{xx}(\omega)$ and $S_{yy}(\omega) = |H(j\omega)|^2\,S_{xx}(\omega)$, where S is the energy (or power) spectral density.

Fourier series

EXPONENTIAL FOURIER SERIES (SYNTHESIS AND ANALYSIS)

$$x(t) = \sum_{k=-\infty}^{\infty} c_k\,e^{jk\omega_0 t}$$

$$c_k = \frac{1}{T_0}\int_{T_0} x(t)\,e^{-jk\omega_0 t}\,dt$$

Holds for real or complex $x(t)$. c_0 is the average (DC) value. Plotting $|c_k|$ and $\angle c_k$ against $k\omega_0$ gives the two-sided line spectrum.

TRIGONOMETRIC FOURIER SERIES

$$x(t) = a_0 + \sum_{k=1}^{\infty}(a_k\cos k\omega_0 t + b_k \sin k\omega_0 t)$$

For real $x(t)$, with a_0 the DC value (convention above).

TRIGONOMETRIC COEFFICIENTS

$$a_0 = \frac{1}{T_0}\int_{T_0} x(t)\,dt$$

$$a_k = \frac{2}{T_0}\int_{T_0} x(t)\cos k\omega_0 t\,dt$$

$$b_k = \frac{2}{T_0}\int_{T_0} x(t)\sin k\omega_0 t\,dt$$

Here $k = 1, 2, 3, \ldots$; the factor is $2/T_0$ for a_k and b_k but $1/T_0$ for a_0.

COMPACT (AMPLITUDE-PHASE) FORM

$$x(t) = C_0 + \sum_{k=1}^{\infty} C_k \cos\left(k\omega_0 t + \theta_k\right), \qquad C_k \geq 0$$

For real $x(t)$. This is the one-sided spectrum: C_k is the physical peak amplitude of the k-th harmonic, and θ_k its phase.

RELATIONS BETWEEN THE COEFFICIENTS

$$c_0 = a_0 = C_0$$
$$c_k = \frac{a_k - jb_k}{2} = \frac{C_k}{2} e^{j\theta_k}, \qquad c_{-k} = c_k^*$$
$$a_k = 2\,\mathrm{Re}\{c_k\}, \qquad b_k = -2\,\mathrm{Im}\{c_k\}$$
$$C_k = \sqrt{a_k^2 + b_k^2} = 2|c_k|, \qquad \theta_k = \angle c_k$$

For real $x(t)$ and $k \geq 1$. θ_k is the angle of the phasor $a_k - jb_k$, so $\cos\theta_k = a_k/C_k$ and $\sin\theta_k = -b_k/C_k$. The shortcut $\theta_k = -\tan^{-1}(b_k/a_k)$ is right only when $a_k > 0$; when $a_k < 0$, add or subtract π.

VALUE OF THE SERIES AT A JUMP

$$\lim_{N\to\infty} \sum_{k=-N}^{N} c_k\, e^{jk\omega_0 t_0} = \frac{x(t_0^+) + x(t_0^-)}{2}$$

When the Dirichlet conditions hold, the series equals $x(t)$ wherever x is continuous and the midpoint of the jump at a discontinuity t_0. The partial sums are taken symmetrically, $k = -N$ to N. A square wave switching between 0 and 5 has series value 2.5 at every switching instant.

HALF-PERIOD COEFFICIENT FORMULAS

$$\text{even: } a_k = \frac{4}{T_0}\int_0^{T_0/2} x(t)\cos k\omega_0 t\, dt$$
$$\text{odd: } b_k = \frac{4}{T_0}\int_0^{T_0/2} x(t)\sin k\omega_0 t\, dt$$
$$\text{half-wave, odd } k\text{: } a_k = \frac{4}{T_0}\int_0^{T_0/2} x(t)\cos k\omega_0 t\, dt$$
$$b_k = \frac{4}{T_0}\int_0^{T_0/2} x(t)\sin k\omega_0 t\, dt$$

For an even signal the DC value is also $a_0 = \frac{2}{T_0}\int_0^{T_0/2} x(t)\,dt$. Under half-wave symmetry every even-k coefficient is zero, so these formulas are used for odd k only.

TIME SHIFT AND TIME REVERSAL

$$x(t - t_0) \leftrightarrow c_k\, e^{-jk\omega_0 t_0}, \qquad x(-t) \leftrightarrow c_{-k}$$

A shift leaves every $|c_k|$ and the power unchanged. A shift of half a period gives $c_k e^{-jk\pi} = (-1)^k c_k$.

DIFFERENTIATION AND INTEGRATION

$$\frac{dx}{dt} \leftrightarrow jk\omega_0\, c_k$$
$$\int_0^t x(\tau)\,d\tau \leftrightarrow \frac{c_k}{jk\omega_0} \quad (k \neq 0,\ c_0 = 0)$$

Differentiating removes the DC and boosts high harmonics; integrating smooths. The integration rule needs zero DC in $x(t)$. Any other running integral (textbooks often write the lower limit as $-\infty$) differs from this one only by a constant, so the DC of the result must be found from the waveform.

PARSEVAL'S POWER THEOREM

$$P = \frac{1}{T_0}\int_{T_0} |x(t)|^2\,dt = \sum_{k=-\infty}^{\infty} |c_k|^2$$
$$= a_0^2 + \frac{1}{2}\sum_{k=1}^{\infty}\left(a_k^2 + b_k^2\right) = C_0^2 + \sum_{k=1}^{\infty}\frac{C_k^2}{2}$$

The trigonometric forms need real $x(t)$. A DC level a_0 carries power a_0^2 and a sinusoid $A\cos(k\omega_0 t + \phi)$ carries $A^2/2$. The RMS value is $\sqrt{P}$.

TOTAL HARMONIC DISTORTION

$$\mathrm{THD} = \frac{\sqrt{\sum_{k=2}^{\infty} C_k^2}}{C_1} = \sqrt{\frac{P - C_0^2}{C_1^2/2} - 1}$$

RMS of all harmonics above the fundamental divided by the RMS of the fundamental (DC excluded). For a symmetric square wave, $\mathrm{THD} = \sqrt{\pi^2/8 - 1} \approx 0.483$.

COEFFICIENTS FROM THE FOURIER TRANSFORM OF ONE PERIOD

$$x(t) = \sum_{n=-\infty}^{\infty} x_p(t - nT_0) \;\Rightarrow\; c_k = \frac{1}{T_0}\,X_p(jk\omega_0)$$
$$X(j\omega) = 2\pi\sum_{k=-\infty}^{\infty} c_k\,\delta(\omega - k\omega_0)$$

$x_p(t)$ is one period of $x(t)$ (zero outside that period) and $X_p(j\omega) = \int_{-\infty}^{\infty} x_p(t)e^{-j\omega t}\,dt$ is its Fourier transform in ω (rad/s). The series coefficients are samples of the single pulse's transform at the harmonics, scaled by $1/T_0$. The transform of the periodic signal itself is a train of impulses of area $2\pi c_k$ at $\omega = k\omega_0$; in the Hz convention it is $X(f) = \sum_k c_k\,\delta(f - kf_0)$, with $c_k = \frac{1}{T_0}X_p(kf_0)$. Example: a rectangular pulse of height A and width τ has $X_p(j\omega) = A\tau\,\mathrm{sinc}\left(\frac{\omega\tau}{2\pi}\right)$, which gives the pulse-train coefficients below.

SQUARE WAVE (ODD, AMPLITUDE A)

$$x(t) = \frac{4A}{\pi}\sum_{k\ \text{odd}}\frac{1}{k}\sin k\omega_0 t$$
$$c_k = \frac{2A}{j\pi k}\ (k\ \text{odd}), \qquad c_k = 0\ (k\ \text{even})$$

$x = +A$ on $(0, T_0/2)$ and $-A$ on $(-T_0/2, 0)$: odd and half-wave symmetric. Power A^2, of which the fundamental carries $8/\pi^2 \approx 0.81$. Shifted to be even ($+A$ for $|t| < T_0/4$): $\frac{4A}{\pi}\left(\cos\omega_0 t - \frac{1}{3}\cos 3\omega_0 t + \frac{1}{5}\cos 5\omega_0 t - \cdots\right)$.

RECTANGULAR PULSE TRAIN

$$c_k = \frac{A\tau}{T_0}\,\mathrm{sinc}\left(\frac{k\tau}{T_0}\right) = \frac{A}{k\pi}\sin\left(\frac{k\pi\tau}{T_0}\right)$$

Height A, width τ, centred on $t = 0$ (even, so c_k is real); the second form is for $k \neq 0$. With duty cycle $d = \tau/T_0$: DC $= Ad$, power $= A^2 d$, and $c_k = 0$ whenever kd is a nonzero integer. At $d = 1/2$ the even harmonics vanish; at $d = 1/3$ every third harmonic vanishes.

TRIANGULAR WAVE (EVEN, ZERO MEAN)

$$x(t) = \frac{8A}{\pi^2}\sum_{k\ \text{odd}}\frac{1}{k^2}\cos k\omega_0 t$$
$$c_k = \frac{4A}{\pi^2 k^2}\ (k\ \text{odd}), \qquad c_k = 0\ (k\ \text{even})$$

$x(t) = A\,(1 - 4|t|/T_0)$ for $|t| \le T_0/2$: $+A$ at $t = 0$, $-A$ at $t = \pm T_0/2$. Even and half-wave symmetric; power $A^2/3$. The unipolar version between 0 and A is $\frac{A}{2} + \frac{4A}{\pi^2}\sum_{k\ \text{odd}}\frac{1}{k^2}\cos k\omega_0 t$.

SAWTOOTH WAVE (ODD)

$$x(t) = \frac{2A}{\pi}\sum_{k=1}^{\infty}\frac{(-1)^{k+1}}{k}\sin k\omega_0 t$$

$$c_k = \frac{jA(-1)^k}{\pi k}\ (k \neq 0), \qquad c_0 = 0$$

$x(t) = 2At/T_0$ on $(-T_0/2, T_0/2)$, rising from $-A$ to $+A$ and jumping back. Odd but not half-wave symmetric, so every harmonic is present; power $A^2/3$. The ramp $x(t) = At/T_0$ on $(0, T_0)$ is $\frac{A}{2} - \frac{A}{\pi}\sum_{k=1}^{\infty}\frac{1}{k}\sin k\omega_0 t$.

PERIODIC IMPULSE TRAIN

$$\sum_{n=-\infty}^{\infty}\delta(t - nT_0) = \frac{1}{T_0}\sum_{k=-\infty}^{\infty}e^{jk\omega_0 t}$$

$$= \frac{1}{T_0} + \frac{2}{T_0}\sum_{k=1}^{\infty}\cos k\omega_0 t$$

$c_k = 1/T_0$ for every k: a flat line spectrum that never decays. Impulses of area I at $t = nT_0$ give $c_k = I/T_0$. The average power is infinite.

FULL-WAVE RECTIFIED SINE

$$A|\sin\omega_1 t| = \frac{2A}{\pi} - \frac{4A}{\pi}\sum_{k=1}^{\infty}\frac{\cos 2k\omega_1 t}{4k^2 - 1}$$

$$A|\cos\omega_1 t| = \frac{2A}{\pi} + \frac{4A}{\pi}\sum_{k=1}^{\infty}\frac{(-1)^{k+1}\cos 2k\omega_1 t}{4k^2 - 1}$$

ω_1 is the frequency of the sine before rectification. The period halves to π/ω_1, so the fundamental is $\omega_0 = 2\omega_1$ and only even multiples of ω_1 appear. DC $= 2A/\pi$; RMS $= A/\sqrt{2}$ (power $A^2/2$, the same as the unrectified sine).

HALF-WAVE RECTIFIED SINE

$$A\max(\sin\omega_0 t,\ 0) = \frac{A}{\pi} + \frac{A}{2}\sin\omega_0 t$$

$$- \frac{2A}{\pi}\sum_{k=1}^{\infty}\frac{\cos 2k\omega_0 t}{4k^2 - 1}$$

The period is unchanged. DC $= A/\pi$, fundamental amplitude $A/2$, then even harmonics only. RMS $= A/2$ (power $A^2/4$). It equals half of (the sine plus its full-wave rectified version).

COEFFICIENT DECAY RATE

$$x, x', \ldots, x^{(m-1)} \text{ continuous},\ x^{(m)} \text{ has jumps}$$

$$\Rightarrow\ |c_k| \text{ decays as } 1/k^{m+1}$$

For a piecewise-smooth periodic $x(t)$. "Decays as" describes the envelope of $|c_k|$; individual coefficients can still be zero, as in a pulse train. Each differentiation multiplies c_k by $jk\omega_0$. A jump in x itself ($m = 0$) gives $1/k$; a kink ($m = 1$) gives $1/k^2$; impulses in x give no decay. The smoother the waveform, the faster its spectrum falls and the fewer terms a good approximation needs.

PERIODIC INPUT TO AN LTI SYSTEM

$$x(t) = \sum_k c_k\, e^{jk\omega_0 t}$$

$$y(t) = \sum_k c_k\, H(jk\omega_0)\, e^{jk\omega_0 t}$$

$$d_k = H(jk\omega_0)\, c_k$$

Steady state of a BIBO-stable LTI system, after any switch-on transient has died out. $H(j\omega)$ is needed only at the harmonic frequencies $k\omega_0$.

SINUSOID THROUGH AN LTI SYSTEM (REAL FORM)

$$A\cos(k\omega_0 t+\theta) \;\rightarrow\; A\,|H(jk\omega_0)|\cos\left(k\omega_0 t+\theta+\angle H(jk\omega_0)\right)$$

For real $h(t)$, so that $H(-j\omega)=H^*(j\omega)$. DC passes with gain $H(0)$. Output power by Parseval: $P_y=\sum_k |H(jk\omega_0)|^2\,|c_k|^2$.

DTFS SYNTHESIS AND ANALYSIS

$$x[n]=\sum_{k=\langle N\rangle} a_k\, e^{jk\Omega_0 n}$$

$$a_k=\frac{1}{N}\sum_{n=\langle N\rangle} x[n]\, e^{-jk\Omega_0 n}, \qquad \Omega_0=\frac{2\pi}{N}$$

$\langle N\rangle$ means any N consecutive integers, e.g. 0 to $N-1$. The coefficients are periodic, $a_{k+N}=a_k$. For real $x[n]$, $a_{-k}=a_{N-k}=a_k^*$. a_0 is the average over one period.

DTFS PARSEVAL AND THE DFT

$$\frac{1}{N}\sum_{n=\langle N\rangle}|x[n]|^2=\sum_{k=\langle N\rangle}|a_k|^2$$

$$X[k]=\sum_{n=0}^{N-1} x[n]\,W_N^{kn}=Na_k, \qquad W_N=e^{-j2\pi/N}$$

The N-point DFT of one period is N times the DTFS coefficients. Time shift: $x[n-n_0]\leftrightarrow a_k e^{-jk\Omega_0 n_0}$. Through a stable LTI system: $a_k\rightarrow a_k H(e^{jk\Omega_0})$, where $H(e^{j\Omega})$ is the DTFT of $h[n]$.

STANDARD DTFS PAIRS

$$\sum_{r=-\infty}^{\infty}\delta[n-rN]\leftrightarrow a_k=\frac{1}{N}\quad\text{for every } k$$

$$\text{rectangular pulse, } |n|\le N_1\leftrightarrow a_k=\frac{1}{N}\,\frac{\sin\left(\pi k(2N_1+1)/N\right)}{\sin\left(\pi k/N\right)}$$

The pulse is $x[n]=1$ for $|n|\le N_1$ and 0 for the rest of the period, repeated every N samples ($2N_1+1<N$). It is even, so a_k is real. The ratio-of-sines form is for $k\neq 0,\pm N,\pm 2N,\ldots$; at those k, $a_k=(2N_1+1)/N$, the average. The ratio of sines is the discrete counterpart of the sinc envelope of a continuous pulse train.

Fourier transform, frequency response, group and phase delay

FOURIER TRANSFORM PAIR (Ω FORM, USED IN THESE NOTES)

$$X(\omega)=\int_{-\infty}^{\infty} x(t)\,e^{-j\omega t}\,dt \qquad x(t)=\frac{1}{2\pi}\int_{-\infty}^{\infty} X(\omega)\,e^{j\omega t}\,d\omega$$

ω in rad/s. The $1/2\pi$ sits only in the inverse, so it reappears in duality, the multiplication property, Parseval's theorem and the formula for $x(0)$.

FOURIER TRANSFORM PAIR (F FORM)

$$X(f)=\int_{-\infty}^{\infty} x(t)\,e^{-j2\pi ft}\,dt \qquad x(t)=\int_{-\infty}^{\infty} X(f)\,e^{j2\pi ft}\,df$$

f in Hz; symmetric, with no factor outside the integrals. Convert with $X(f)=X(\omega)\big|_{\omega=2\pi f}$, and for impulses use $\delta(\omega)=\delta(f)/2\pi$, so $2\pi\delta(\omega)$ becomes $\delta(f)$.

SINC CONVENTION

$$\operatorname{sinc}(x) = \frac{\sin(\pi x)}{\pi x}, \qquad \operatorname{sinc}(0) = 1$$

Normalised sinc, with zeros at $x = \pm 1, \pm 2, \ldots$. Some texts use $\sin x / x$ under the same name; every sinc result below is also written in explicit $\sin$ form, so nothing depends on the convention.

LINK WITH THE LAPLACE TRANSFORM

$$X(\omega) = X(s)\big|_{s=j\omega} \quad \text{only if the ROC of } X(s) \text{ contains the } j\omega \text{ axis}$$

$X(s)$ is the bilateral Laplace transform. Example: $e^{-at}u(t)$, $a > 0$, has ROC $\operatorname{Re}(s) > -a$, which contains the axis. If the ROC boundary is the $j\omega$ axis (poles on the axis, as for $u(t)$ with ROC $\operatorname{Re}(s) > 0$), the Fourier transform exists only with impulses and the substitution misses them. If the ROC excludes the axis altogether (as for $e^{2t}u(t)$, ROC $\operatorname{Re}(s) > 2$), there is no Fourier transform.

RECTANGULAR PULSE AND SINC

$$\operatorname{rect}\left(\frac{t}{T}\right) \leftrightarrow \frac{2\sin(\omega T/2)}{\omega} = T \operatorname{sinc}\left(\frac{\omega T}{2\pi}\right) = T \operatorname{sinc}(fT)$$

Pulse of width T and height 1. $X(0) = T$, the pulse area. First nulls at $\omega = \pm 2\pi/T$, i.e. $f = \pm 1/T$: a narrower pulse has a wider main lobe.

IDEAL LOWPASS SPECTRUM AND ITS TIME FUNCTION

$$\frac{\sin Wt}{\pi t} = \frac{W}{\pi} \operatorname{sinc}\left(\frac{Wt}{\pi}\right) \leftrightarrow \begin{cases} 1, & |\omega| < W \\ 0, & |\omega| > W \end{cases}$$

The dual of the rect–sinc pair. Peak value W/π at $t = 0$, zeros at $t = k\pi/W$ ($k \neq 0$). This is $h(t)$ of the ideal lowpass filter with cutoff W rad/s.

ONE-SIDED AND TWO-SIDED EXPONENTIALS

$$e^{-at}u(t) \leftrightarrow \frac{1}{a + j\omega}, \qquad e^{-a|t|} \leftrightarrow \frac{2a}{a^2 + \omega^2} \qquad (a > 0)$$

$e^{-a|t|} = e^{-at}u(t) + e^{at}u(-t)$, so its transform is $\dfrac{1}{a + j\omega} + \dfrac{1}{a - j\omega}$. For $a < 0$ both signals grow and have no Fourier transform; $a = 0$ gives $u(t)$ or 1, which need impulses.

UNIT STEP AND SIGNUM

$$u(t) \leftrightarrow \pi\delta(\omega) + \frac{1}{j\omega}, \qquad \operatorname{sgn}(t) \leftrightarrow \frac{2}{j\omega}$$

The impulse in $\mathcal{F}\{u(t)\}$ is the transform of its average value: $\frac{1}{2} \cdot 2\pi\delta(\omega) = \pi\delta(\omega)$. $\operatorname{sgn}(t)$ has zero average, so it has no impulse.

GAUSSIAN

$$e^{-at^2} \leftrightarrow \sqrt{\frac{\pi}{a}}\, e^{-\omega^2/(4a)} \qquad (a > 0)$$

A narrow Gaussian in time (large a) gives a wide Gaussian in frequency. In the f form, $e^{-\pi t^2} \leftrightarrow e^{-\pi f^2}$ exactly.

PERIODIC IMPULSE TRAIN

$$\sum_{n=-\infty}^{\infty} \delta(t - nT) \leftrightarrow \omega_s \sum_{k=-\infty}^{\infty} \delta(\omega - k\omega_s), \qquad \omega_s = \frac{2\pi}{T}$$

Every Fourier-series coefficient of the impulse train equals $1/T$, which gives impulses of area $2\pi/T$. This pair is the basis of sampling theory.

TIME SCALING (WITH SHIFT)

$$x(at) \leftrightarrow \frac{1}{|a|} X\left(\frac{\omega}{a}\right), \qquad x(at-b) \leftrightarrow \frac{1}{|a|} e^{-j\omega b/a} X\left(\frac{\omega}{a}\right)$$

Real $a \neq 0$. Compressing in time ($|a| > 1$) stretches the spectrum and lowers it. Since $x(at-b) = x\big(a(t-b/a)\big)$, the delay is b/a, not b.

DUALITY

$$\text{if } x(t) \leftrightarrow X(\omega) \text{ then } X(t) \leftrightarrow 2\pi\, x(-\omega)$$

In the f form there is no 2π: $X(t) \leftrightarrow x(-f)$. For an even x, $x(-\omega) = x(\omega)$.

MODULATION

$$x(t)\cos\omega_0 t \leftrightarrow \tfrac{1}{2}\left[X(\omega-\omega_0) + X(\omega+\omega_0)\right]$$

Each shifted copy has half the height. With $\sin\omega_0 t$ instead: $\dfrac{1}{2j}\left[X(\omega-\omega_0) - X(\omega+\omega_0)\right]$. If the copies do not overlap, the energy is halved.

DIFFERENTIATION IN TIME AND IN FREQUENCY

$$\frac{dx}{dt} \leftrightarrow j\omega X(\omega), \qquad t\,x(t) \leftrightarrow j\frac{dX(\omega)}{d\omega}$$

Differentiation removes any DC impulse: $\dfrac{d}{dt}u(t) = \delta(t)$, and $j\omega\left[\pi\delta(\omega) + \dfrac{1}{j\omega}\right] = 1$ because $\omega\,\delta(\omega) = 0$.

INTEGRATION

$$\int_{-\infty}^{t} x(\tau)\,d\tau \leftrightarrow \frac{X(\omega)}{j\omega} + \pi X(0)\,\delta(\omega)$$

The impulse term vanishes only when $X(0) = \int x\,dt = 0$. Dividing by $j\omega$ alone is the classic error.

CONVOLUTION AND MULTIPLICATION

$$x(t) * y(t) \leftrightarrow X(\omega)Y(\omega), \qquad x(t)\,y(t) \leftrightarrow \frac{1}{2\pi}\,X(\omega) * Y(\omega)$$

The $1/2\pi$ belongs to the multiplication property in the ω form only. The convolution property is why an LTI output is $Y(\omega) = H(\omega)X(\omega)$.

AREA AND VALUE THEOREMS

$$X(0) = \int_{-\infty}^{\infty} x(t)\,dt, \qquad x(0) = \frac{1}{2\pi}\int_{-\infty}^{\infty} X(\omega)\,d\omega$$

Put $\omega = 0$ in the forward integral, or $t = 0$ in the inverse. In the f form, $x(0) = \int X(f)\,df$. If x jumps at $t = 0$, the second integral gives the midpoint of the jump.

PARSEVAL / RAYLEIGH ENERGY THEOREM

$$E = \int_{-\infty}^{\infty} |x(t)|^2\,dt = \frac{1}{2\pi}\int_{-\infty}^{\infty} |X(\omega)|^2\,d\omega = \int_{-\infty}^{\infty} |X(f)|^2\,df$$

For energy signals. Generalised form: $\int x(t)\,y^*(t)\,dt = \dfrac{1}{2\pi}\int X(\omega)\,Y^*(\omega)\,d\omega$.

ENERGY SPECTRAL DENSITY (ESD) AND ENERGY IN A BAND

$$\Psi_x(\omega) = |X(\omega)|^2, \qquad E_{\omega_1 \le |\omega| \le \omega_2} = \frac{1}{2\pi}\cdot 2\int_{\omega_1}^{\omega_2} |X(\omega)|^2\,d\omega$$

The band formula is for real $x(t)$. The factor 2 counts the matching negative-frequency band, which carries the same energy because $|X|$ is even.

ESD THROUGH AN LTI SYSTEM

$$\Psi_y(\omega) = |H(\omega)|^2\,\Psi_x(\omega), \qquad E_y = \frac{1}{2\pi}\int_{-\infty}^{\infty} |H(\omega)|^2 |X(\omega)|^2\,d\omega$$

The phase of H has no effect on output energy.

FOURIER TRANSFORM OF A PERIODIC SIGNAL

$$x(t) = \sum_{k=-\infty}^{\infty} c_k\, e^{jk\omega_0 t} \;\leftrightarrow\; X(\omega) = 2\pi \sum_{k=-\infty}^{\infty} c_k\,\delta(\omega - k\omega_0)$$

$\omega_0 = 2\pi/T_0$. In the f form, $X(f) = \sum_k c_k\,\delta(f - kf_0)$ with $f_0 = 1/T_0$.

FOURIER-SERIES COEFFICIENTS FROM THE TRANSFORM OF ONE PERIOD

$$c_k = \frac{1}{T_0}\,X_1(\omega)\Big|_{\omega = k\omega_0}$$

$X_1(\omega)$ is the transform of one period of $x(t)$ (taken as zero outside that period). The coefficients are samples of $X_1(\omega)/T_0$.

FREQUENCY RESPONSE

$$H(j\omega) = \int_{-\infty}^{\infty} h(t)\,e^{-j\omega t}\,dt = H(s)\big|_{s=j\omega}, \qquad Y(\omega) = H(\omega)\,X(\omega)$$

Putting $s = j\omega$ is valid when the system is BIBO stable ($\int |h|\,dt < \infty$), so the ROC contains the $j\omega$ axis. For a stable system described by $\sum a_k y^{(k)} = \sum b_k x^{(k)}$, $H(j\omega) = \dfrac{\sum b_k (j\omega)^k}{\sum a_k (j\omega)^k}$.

SINUSOIDAL STEADY STATE

$$A\cos(\omega_0 t + \phi) \;\longrightarrow\; A\,|H(j\omega_0)|\cos\big(\omega_0 t + \phi + \theta(\omega_0)\big)$$

Real $h(t)$, stable system, input applied for all time. The frequency is unchanged; only amplitude and phase change. Superpose for several tones.

DISTORTIONLESS SYSTEM

$$H(j\omega) = K\,e^{-j\omega t_d}: \qquad |H(j\omega)| = K, \quad \theta(\omega) = -\omega t_d$$

Here $K > 0$. The conditions are needed only across the band the signal occupies. Adding a multiple of 2π to the phase changes nothing, and a phase of $-\omega t_d \pm \pi$ gives $y(t) = -Kx(t - t_d)$, an inverted but undistorted copy. Any other constant phase offset changes the waveform (for a narrowband bandpass signal it shifts only the carrier phase; see below).

PHASE DELAY AND GROUP DELAY

$$\tau_p(\omega) = -\frac{\theta(\omega)}{\omega}, \qquad \tau_g(\omega) = -\frac{d\theta(\omega)}{d\omega}$$

θ in radians and ω in rad/s give seconds. Use the unwrapped phase for τ_p. For $\theta = -\omega t_d$, $\tau_p = \tau_g = t_d$ at every frequency; conversely, $\tau_p = \tau_g$ at every frequency only when $\theta = -\omega t_d$ for some constant t_d. In terms of f: $\tau_p = -\dfrac{\theta}{2\pi f}$ and $\tau_g = -\dfrac{1}{2\pi}\dfrac{d\theta}{df}$.

NARROWBAND OUTPUT: ENVELOPE AND CARRIER DELAYS

$$y(t) \approx |H(j\omega_c)|\,m(t - \tau_g)\cos\big(\omega_c(t - \tau_p)\big), \qquad \tau_g = \tau_g(\omega_c),\ \tau_p = \tau_p(\omega_c)$$

The envelope (the information) is delayed by the group delay, also called envelope delay. The carrier is delayed by the phase delay. If τ_g varies across the band, the envelope is dispersed.

DISCRETE-TIME GROUP DELAY; LINEAR-PHASE FIR

$$\tau_g(\Omega) = -\frac{d\theta(\Omega)}{d\Omega} \text{ samples}; \qquad h[n] = h[N-1-n] \;\Rightarrow\; \tau_g = \frac{N-1}{2}$$

A length-N FIR filter with symmetric coefficients has $H(e^{j\Omega}) = e^{-j\Omega(N-1)/2}A(\Omega)$ with A real, so its phase is linear (apart from jumps of π where A changes sign). Antisymmetric coefficients, $h[n] = -h[N-1-n]$, give $H(e^{j\Omega}) = j\,e^{-j\Omega(N-1)/2}A(\Omega)$: the same group delay $(N-1)/2$ with a constant $\pi/2$ phase offset. Example: $h[n] = \{1, 2, 1\}$ for $n = 0, 1, 2$ gives $e^{-j\Omega}(2 + 2\cos\Omega)$, a delay of 1 sample.

3-DB BANDWIDTH OF A FIRST-ORDER LOWPASS

$$H(j\omega) = \frac{1}{1 + j\omega RC}: \qquad |H| = \frac{1}{\sqrt{2}} \text{ at } \omega_{3\text{dB}} = \frac{1}{RC} \text{ rad/s} \left(f_{3\text{dB}} = \frac{1}{2\pi RC} \text{ Hz}\right)$$

At the 3-dB point the phase is -45°. The 10–90% step rise time is $t_r = RC\ln 9 \approx 2.2RC$, so $t_r\, f_{3\text{dB}} \approx 0.35$.

HILBERT TRANSFORM

$$\hat{x}(t) = x(t) * \frac{1}{\pi t}, \qquad H(j\omega) = -j\,\text{sgn}(\omega) = \begin{cases} -j, & \omega > 0 \\ +j, & \omega < 0 \end{cases}$$

Gain 1 at every $\omega \neq 0$, phase -90° for $\omega > 0$. Its $h(t) = 1/(\pi t)$ is non-causal, so the ideal Hilbert transformer is also unrealisable.

DTFT PAIR

$$X(e^{j\Omega}) = \sum_{n=-\infty}^{\infty} x[n]\, e^{-j\Omega n} \qquad x[n] = \frac{1}{2\pi}\int_{-\pi}^{\pi} X(e^{j\Omega})\, e^{j\Omega n}\, d\Omega$$

Any interval of length 2π may replace $-\pi$ to π. The DTFT is the z-transform on the unit circle, $X(e^{j\Omega}) = X(z)\big|_{z=e^{j\Omega}}$, valid when the ROC of $X(z)$ contains $|z| = 1$.

DTFT VALUE THEOREMS

$$X(e^{j0}) = \sum_n x[n], \qquad X(e^{j\pi}) = \sum_n (-1)^n x[n], \qquad x[0] = \frac{1}{2\pi}\int_{-\pi}^{\pi} X(e^{j\Omega})\, d\Omega$$

Put $\Omega = 0$ or $\Omega = \pi$ in the forward sum, or $n = 0$ in the inverse integral. More generally $\frac{1}{2\pi}\int_{-\pi}^{\pi} X(e^{j\Omega})\, e^{j\Omega k}\, d\Omega = x[k]$.

PARSEVAL'S THEOREM FOR THE DTFT

$$\sum_{n=-\infty}^{\infty} |x[n]|^2 = \frac{1}{2\pi}\int_{-\pi}^{\pi} |X(e^{j\Omega})|^2\, d\Omega$$

The frequency integral spans one period, 2π wide, never $-\infty$ to ∞.

DISCRETE-TIME SINUSOIDAL STEADY STATE

$$A\cos(\Omega_0 n + \phi) \;\longrightarrow\; A\,|H(e^{j\Omega_0})|\cos\left(\Omega_0 n + \phi + \theta(\Omega_0)\right)$$

Real $h[n]$, stable system ($\sum |h[n]| < \infty$), input applied for all n. Ω_0 and $\Omega_0 + 2\pi$ give the same sequence and the same response.

Laplace transform, poles, zeros and system functions

BILATERAL LAPLACE TRANSFORM

$$X(s) = \mathcal{L}\{x(t)\} = \int_{-\infty}^{\infty} x(t)\, e^{-st}\, dt, \qquad s = \sigma + j\omega$$

Converges only for s in the ROC, which must be quoted with $X(s)$.

UNILATERAL LAPLACE TRANSFORM

$$X(s) = \int_{0^-}^{\infty} x(t)\, e^{-st}\, dt$$

Lower limit 0^-, so an impulse at $t = 0$ is included. It equals the bilateral transform of the part of $x(t)$ from $t = 0^-$ onwards, so, when it exists, its ROC is a right half-plane $\mathrm{Re}(s) > \sigma_0$ (the whole plane if that part has finite duration) and is usually not written. For a causal signal the two transforms are identical. This is the version used for differential equations with initial conditions.

LINK TO THE FOURIER TRANSFORM

$$X(j\omega) = X(s)\big|_{s=j\omega} \qquad (\text{only if the ROC contains the } j\omega \text{ axis})$$

In the Hz form, $X(f) = X(s)\big|_{s=j2\pi f}$. If a pole lies on the $j\omega$ axis (as for $u(t)$) the substitution is not valid.

IMPULSE, STEP, RAMP AND POWERS OF T

$$\delta(t) \leftrightarrow 1 \;\; (\text{all } s); \qquad u(t) \leftrightarrow \frac{1}{s}, \quad t\,u(t) \leftrightarrow \frac{1}{s^2}, \quad \frac{t^n}{n!}\,u(t) \leftrightarrow \frac{1}{s^{n+1}} \;\; (\text{each } \mathrm{Re}(s) > 0)$$

$n = 0, 1, 2, \ldots$ Also, for all s: $\delta(t - t_0) \leftrightarrow e^{-st_0}$, and the derivatives of the impulse $\frac{d^n\delta(t)}{dt^n} \leftrightarrow s^n$ (so $\delta'(t) \leftrightarrow s$).

EXPONENTIALS

$$e^{-at}u(t) \leftrightarrow \frac{1}{s+a}, \qquad \frac{t^n}{n!}\,e^{-at}u(t) \leftrightarrow \frac{1}{(s+a)^{n+1}} \qquad (\mathrm{Re}(s) > -\mathrm{Re}(a))$$

Valid for real or complex a. The left-sided versions are $-e^{-at}u(-t)$ and $-\frac{t^n}{n!}e^{-at}u(-t)$, with ROC $\mathrm{Re}(s) < -\mathrm{Re}(a)$. For real a the hyperbolic pairs follow: $\cosh(at)\,u(t) \leftrightarrow \frac{s}{s^2 - a^2}$ and $\sinh(at)\,u(t) \leftrightarrow \frac{a}{s^2 - a^2}$, both with ROC $\mathrm{Re}(s) > |a|$.

SINUSOIDS

$$\cos(\omega_0 t)\,u(t) \leftrightarrow \frac{s}{s^2 + \omega_0^2}, \qquad \sin(\omega_0 t)\,u(t) \leftrightarrow \frac{\omega_0}{s^2 + \omega_0^2} \qquad (\mathrm{Re}(s) > 0)$$

ω_0 real. The poles at $\pm j\omega_0$ lie on the $j\omega$ axis, which the ROC does not contain, so putting $s = j\omega$ does not give the Fourier transform of these one-sided sinusoids (theirs contains impulses at $\omega = \pm\omega_0$).

SINUSOIDS MULTIPLIED BY T (DOUBLE POLES ON THE JΩ AXIS)

$$t\cos(\omega_0 t)\,u(t) \leftrightarrow \frac{s^2 - \omega_0^2}{(s^2 + \omega_0^2)^2}, \qquad t\sin(\omega_0 t)\,u(t) \leftrightarrow \frac{2\omega_0 s}{(s^2 + \omega_0^2)^2} \qquad (\mathrm{Re}(s) > 0)$$

From the sinusoid pairs by multiplication by t. Combining them gives $\frac{1}{(s^2 + \omega_0^2)^2} \leftrightarrow \frac{\sin(\omega_0 t) - \omega_0 t\cos(\omega_0 t)}{2\omega_0^3}\,u(t)$: a double pole on the $j\omega$ axis gives a term that grows like t.

DAMPED SINUSOIDS

$$e^{-at}\cos(\omega_0 t)\,u(t) \leftrightarrow \frac{s+a}{(s+a)^2 + \omega_0^2}, \qquad e^{-at}\sin(\omega_0 t)\,u(t) \leftrightarrow \frac{\omega_0}{(s+a)^2 + \omega_0^2} \qquad (\mathrm{Re}(s) > -a)$$

a and ω_0 real. Poles at $-a \pm j\omega_0$. These are the undamped pairs with s replaced by $s + a$ (shift in s).

MULTIPLICATION BY POWERS OF T

$$t^n x(t) \leftrightarrow (-1)^n \frac{d^n X(s)}{ds^n}$$

Same ROC as $X(s)$. With $n = 1$ and $x(t) = e^{-at}u(t)$ it gives $t\,e^{-at}u(t) \leftrightarrow \frac{1}{(s+a)^2}$.

UNILATERAL DIFFERENTIATION (BRINGS IN INITIAL CONDITIONS)

$$\frac{dx}{dt} \leftrightarrow sX(s) - x(0^-), \qquad \frac{d^2x}{dt^2} \leftrightarrow s^2X(s) - s\,x(0^-) - x'(0^-)$$

Unilateral transform only. In general $\frac{d^nx}{dt^n} \leftrightarrow s^nX(s) - \sum_{k=1}^{n} s^{n-k}x^{(k-1)}(0^-)$. The values are taken at 0^-, before the input is applied.

UNILATERAL TIME SHIFT (DELAY)

$$x(t-t_0)\,u(t-t_0) \leftrightarrow e^{-st_0}X(s), \qquad t_0 > 0$$

$X(s)$ is the transform of $x(t)u(t)$. The whole causal signal must be delayed; $x(t-t_0)u(t)$ is a different signal and does not simply pick up e^{-st_0}.

INITIAL VALUE THEOREM

$$x(0^+) = \lim_{s\to\infty} s\,X(s)$$

Valid when $x(t) = 0$ for $t < 0$ and $x(t)$ has no impulse (or higher singularity) at $t = 0$; for a rational $X(s)$ this means $X(s)$ is strictly proper (numerator degree less than denominator degree). If the numerator degree is equal or higher, divide first: the polynomial part is the impulse terms at the origin, which do not affect $x(0^+)$, and the theorem applies to the strictly proper remainder.

FINAL VALUE THEOREM

$$\lim_{t\to\infty} x(t) = \lim_{s\to 0} s\,X(s)$$

Valid, for a rational $X(s)$, only if every pole of $sX(s)$ lies in the open LHP; equivalently, every pole of $X(s)$ is in the open LHP except at most a single simple pole at $s = 0$. With a pole in the RHP, a repeated pole at the origin, or any other pole on the $j\omega$ axis, $x(t)$ has no finite limit and the value the formula returns is not the final value.

COEFFICIENT AT A SIMPLE POLE (COVER-UP)

$$X(s) = \sum_k \frac{A_k}{s - p_k}, \qquad A_k = (s - p_k)\,X(s)\Big|_{s=p_k}$$

Each term inverts to $A_k e^{p_k t}u(t)$ if the ROC is to the right of p_k, or to $-A_k e^{p_k t}u(-t)$ if the ROC is to the left of p_k.

COEFFICIENTS AT A POLE OF ORDER R

$$\sum_{m=1}^{r} \frac{B_m}{(s-p)^m}, \qquad B_m = \frac{1}{(r-m)!}\,\frac{d^{r-m}}{ds^{r-m}}\Big[(s-p)^r X(s)\Big]_{s=p}$$

Only the top coefficient B_r is plain cover-up; each lower one needs one more derivative. Inversion: $\frac{1}{(s-p)^m} \leftrightarrow \frac{t^{m-1}}{(m-1)!}e^{pt}u(t)$ for an ROC to the right of p, and $-\frac{t^{m-1}}{(m-1)!}e^{pt}u(-t)$ for an ROC to the left.

COMPLEX-CONJUGATE POLE PAIR (COMPLETE THE SQUARE)

$$\frac{As+B}{(s+a)^2+b^2} = A\,\frac{s+a}{(s+a)^2+b^2} + \frac{B-Aa}{b}\cdot\frac{b}{(s+a)^2+b^2} \;\leftrightarrow\; e^{-at}\left[A\cos bt + \frac{B-Aa}{b}\sin bt\right]u(t)$$

Causal case, ROC $\mathrm{Re}(s) > -a$, with A, B, a, b real and $b > 0$. Residue form of the same result, for a real signal: if the pole $p = -a + jb$ has residue K, the pair contributes $2|K|\,e^{-at}\cos(bt + \angle K)\,u(t)$.

TRANSFER FUNCTION FROM A DIFFERENTIAL EQUATION

$$\sum_{k=0}^{N} a_k \frac{d^k y}{dt^k} = \sum_{k=0}^{M} b_k \frac{d^k x}{dt^k} \implies H(s) = \frac{Y(s)}{X(s)} = \frac{\sum_{k=0}^{M} b_k s^k}{\sum_{k=0}^{N} a_k s^k}$$

Replace each $\frac{d^k}{dt^k}$ by s^k, with zero initial conditions. The equation alone does not fix the ROC; a statement such as causal, stable or initially at rest does.

POLE-ZERO-GAIN FORM

$$H(s) = K\,\frac{(s-z_1)(s-z_2)\cdots(s-z_M)}{(s-p_1)(s-p_2)\cdots(s-p_N)}$$

Zeros z_i are the roots of the numerator and poles p_k the roots of the denominator, after cancelling common factors. $H(s)$ is **proper** if $M \le N$ and **strictly proper** if $M < N$. On a pole-zero plot poles are marked $\times$ and zeros $\circ$. For real $h(t)$ the coefficients are real, so complex poles and zeros come in conjugate pairs. The plot fixes $H(s)$ only up to the gain K.

INTERCONNECTED SYSTEMS

$$\text{cascade: } H_1(s)\,H_2(s), \qquad \text{parallel: } H_1(s)+H_2(s), \qquad \text{negative feedback: } \frac{G(s)}{1+G(s)F(s)}$$

The cascade result assumes the second block does not load the first. In the feedback loop $G(s)$ is the forward path and $F(s)$ the feedback path; with positive feedback the denominator is $1-G(s)F(s)$. Writing $G = N_G/D_G$ and $F = N_F/D_F$, the negative-feedback system is $\dfrac{N_G D_F}{D_G D_F + N_G N_F}$, so its poles are the roots of $D_G D_F + N_G N_F$ (after cancelling any common factor), not the poles of G or F.

SINUSOIDAL STEADY STATE

$$x(t) = A\cos(\omega_0 t + \theta) \implies y(t) = A\,\big|H(j\omega_0)\big|\cos\big(\omega_0 t + \theta + \angle H(j\omega_0)\big)$$

Stable LTI system with real $h(t)$; the input is applied for all time, or $y(t)$ is the steady state after the transients have died out. $H(j\omega) = H(s)\big|_{s=j\omega}$.

GEOMETRIC (VECTOR) EVALUATION

$$\big|H(j\omega)\big| = |K|\,\frac{\prod_i |j\omega - z_i|}{\prod_k |j\omega - p_k|}, \qquad \angle H(j\omega) = \angle K + \sum_i \angle(j\omega - z_i) - \sum_k \angle(j\omega - p_k)$$

$j\omega - z_i$ is the vector from the zero z_i to the point $j\omega$ on the axis; likewise for each pole. Magnitude: product of zero-vector lengths over product of pole-vector lengths. Phase: sum of zero-vector angles minus sum of pole-vector angles, each measured anticlockwise from the positive real direction. $\angle K = 180^\circ$ if $K < 0$.

FIRST-ORDER ALL-PASS

$$H_{\text{ap}}(s) = \frac{a-s}{a+s},\ \ a>0: \qquad \big|H_{\text{ap}}(j\omega)\big| = 1, \qquad \angle H_{\text{ap}}(j\omega) = -2\tan^{-1}\frac{\omega}{a}$$

Pole at $-a$, zero at $+a$. The phase falls from 0 to -180°; the group delay is $\tau(\omega) = -\dfrac{d\angle H}{d\omega} = \dfrac{2a}{a^2+\omega^2}$. The form $\dfrac{s-a}{s+a}$ is the same system times -1, with phase $180^\circ - 2\tan^{-1}(\omega/a)$.

MINIMUM-PHASE CONDITION

$$H(s) \text{ is minimum phase} \iff \text{causal, stable, all poles and all zeros in } \mathrm{Re}(s) < 0$$

Among causal stable systems with the same $|H(j\omega)|$, the minimum-phase one has the least group delay at every ω and the least net phase lag over $0 \le \omega < \infty$, because each all-pass factor has positive group delay and only adds lag. When the numerator and denominator degrees are equal, $1/H(s)$ is also causal and stable. Some texts also allow zeros on the $j\omega$ axis; the open-LHP version is the one used here.

ZERO-INPUT PLUS ZERO-STATE

$$Y(s) = \underbrace{\frac{I(s)}{A(s)}}_{\text{zero-input}} + \underbrace{H(s)\,X(s)}_{\text{zero-state}}, \qquad H(s) = \frac{B(s)}{A(s)}$$

$A(s)$ and $B(s)$ are the polynomials that multiply $Y(s)$ and $X(s)$. $I(s)$ collects the initial-condition terms $y(0^-), y'(0^-), \ldots$ (terms in $x(0^-)$ vanish for a causal input). The natural modes come from the roots of $A(s)$, the forced part from the poles of $X(s)$.

CIRCUIT ELEMENTS IN THE S-DOMAIN

$$V_R(s) = R\,I_R(s), \qquad V_L(s) = sL\,I_L(s) - L\,i_L(0^-), \qquad V_C(s) = \frac{I_C(s)}{sC} + \frac{v_C(0^-)}{s}$$

Unilateral transform of $v = Ri$, $v = L\dfrac{di}{dt}$ and $i = C\dfrac{dv}{dt}$, with the passive sign convention (current entering the $+$ terminal). The terms $-L\,i_L(0^-)$ and $\dfrac{v_C(0^-)}{s}$ act as voltage sources in series with the impedances sL and $\dfrac{1}{sC}$. With zero initial conditions only the impedances R, sL and $\dfrac{1}{sC}$ remain, and a circuit's transfer function follows by ordinary impedance algebra.

Sampling and reconstruction

IMPULSE TRAIN AND ITS TRANSFORM

$$\sum_{n=-\infty}^{\infty} \delta(t - nT_s) \;\leftrightarrow\; f_s \sum_{k=-\infty}^{\infty} \delta(f - kf_s)$$

In ω: the transform is $\omega_s \sum_k \delta(\omega - k\omega_s)$, with $\omega_s = 2\pi/T_s$.

SPECTRUM OF THE IDEALLY SAMPLED SIGNAL

$$X_s(f) = \frac{1}{T_s} \sum_{k=-\infty}^{\infty} X(f - kf_s) = \sum_{n=-\infty}^{\infty} x(nT_s)\, e^{-j2\pi f n T_s}$$

Replicas of $X(f)$ at every multiple of f_s, each scaled by $1/T_s$. In ω: $X_s(\omega) = \dfrac{1}{T_s}\sum_k X(\omega - k\omega_s)$; the scale factor is still $1/T_s$, not ω_s.

SUM OF THE SAMPLES

$$\sum_{n=-\infty}^{\infty} x(nT_s) = \frac{1}{T_s} \sum_{k=-\infty}^{\infty} X(kf_s) = \frac{X(0)}{T_s} = \frac{1}{T_s}\int_{-\infty}^{\infty} x(t)\,dt \qquad (f_m < f_s)$$

Put $f = 0$ in the spectrum formula above. The first equality holds generally; the second needs $X(kf_s) = 0$ for every $k \neq 0$, which a signal band-limited to $f_m < f_s$ guarantees; the rate does not have to reach the Nyquist rate for this. It turns an infinite sum of sinc-type samples into a single transform value.

SAMPLING THEOREM

$$f_s \geq 2f_m \qquad \left(T_s \leq \frac{1}{2f_m}\right)$$

For a signal band-limited to f_m Hz. If $X(f)$ has impulses exactly at $\pm f_m$ (the highest component is a sinusoid), the inequality must be strict: $f_s > 2f_m$.

NYQUIST RATE AND NYQUIST INTERVAL

$$f_N = 2f_m, \qquad T_N = \frac{1}{f_N} = \frac{1}{2f_m}$$

Nyquist rate = twice the HIGHEST frequency present (equal to twice the bandwidth only for a lowpass signal). Nyquist interval = the largest allowed sampling interval. In rad/s: $\omega_N = 2\omega_m$.

BAND LIMITS OF THE SINC FORMS

$$\frac{\sin(2\pi Bt)}{\pi t} = 2B\,\mathrm{sinc}(2Bt) \;\leftrightarrow\; \mathrm{rect}\left(\frac{f}{2B}\right), \qquad \mathrm{sinc}^2(2Bt) \;\leftrightarrow\; \frac{1}{2B}\,\mathrm{tri}\left(\frac{f}{2B}\right)$$

$\mathrm{rect}(f/2B) = 1$ for $|f| < B$; $\mathrm{tri}(f/2B) = 1 - |f|/(2B)$ for $|f| \leq 2B$. So $\sin(2\pi Bt)/(\pi t)$ reaches B Hz and its square reaches $2B$ Hz. In ω: $\sin(Wt)/(\pi t)$ is band-limited to W rad/s $= W/2\pi$ Hz, so its Nyquist rate is W/π samples/s.

APPARENT (ALIASED) FREQUENCY

$$f_a = |f_0 - k f_s|, \qquad k = \text{integer nearest to } \frac{f_0}{f_s}, \qquad 0 \le f_a \le \frac{f_s}{2}$$

The frequency an ideal reconstruction filter of cutoff $f_s/2$ outputs for a tone at f_0 sampled at f_s. If $f_0 < f_s/2$, $k = 0$ and $f_a = f_0$, so there is no aliasing.

SINUSOID SAMPLED AT EXACTLY TWICE ITS FREQUENCY

$$f_s = 2f_0: \qquad x(nT_s) = A\cos(\pi n + \theta) = A(-1)^n \cos\theta$$

The samples depend only on $A\cos\theta$: a sine ($\theta = -\pi/2$) gives all-zero samples, and different (A, θ) pairs give identical samples. A sinusoid therefore needs $f_s > 2f_0$ strictly. In the spectrum, the impulses at $\pm f_0$ fall on each other's replicas at $\pm f_s/2$.

RECONSTRUCTION FILTER AND ALLOWED CUTOFF

$$H_r(f) = T_s \,\mathrm{rect}\left(\frac{f}{2f_c}\right), \qquad f_m < f_c < f_s - f_m$$

Passband gain T_s restores the amplitude. The range exists only when $f_s > 2f_m$; at $f_s = 2f_m$ it shrinks to the single value $f_c = f_s/2$. The endpoints are also acceptable when $X(f)$ has no impulses at $\pm f_m$.

SINC INTERPOLATION FORMULA

$$x(t) = \sum_{n=-\infty}^{\infty} x(nT_s)\,\mathrm{sinc}\left(\frac{t - nT_s}{T_s}\right)$$

Exact for $x(t)$ band-limited to $f_m < f_s/2$, using $f_c = f_s/2$. For any other allowed cutoff: $x(t) = 2f_cT_s \sum_n x(nT_s)\,\mathrm{sinc}\big(2f_c(t - nT_s)\big)$.

NATURAL SAMPLING

$$X_{ns}(f) = \frac{\tau}{T_s} \sum_{k=-\infty}^{\infty} \mathrm{sinc}\left(\frac{k\tau}{T_s}\right) X(f - kf_s)$$

$x(t)$ times a unit-height rectangular pulse train (width τ, period T_s, pulses centred on nT_s). $c_k = (\tau/T_s)\,\mathrm{sinc}(k\tau/T_s)$ are the pulse train's Fourier-series coefficients, one constant per replica. The baseband copy is $(\tau/T_s)X(f)$, undistorted; an ideal LPF with gain T_s/τ recovers $x(t)$.

FLAT-TOP (SAMPLE-AND-HOLD) SAMPLING

$$X_{ft}(f) = \frac{\tau}{T_s}\,\mathrm{sinc}(f\tau)\,e^{-j\pi f\tau} \sum_{k=-\infty}^{\infty} X(f - kf_s)$$

Impulse samples convolved with the hold pulse $\mathrm{rect}\left((t - \tau/2)/\tau\right)$ (width τ, starting at each sampling instant), whose transform is $\tau\,\mathrm{sinc}(f\tau)\,e^{-j\pi f\tau}$. The factor varies with f, so the baseband copy itself is distorted (the **aperture effect**) and delayed by $\tau/2$.

APERTURE-EFFECT ATTENUATION AND EQUALISER

$$A(f) = |\mathrm{sinc}(f\tau)| = \left|\frac{\sin(\pi f\tau)}{\pi f\tau}\right|, \qquad H_{eq}(f) = \frac{1}{\mathrm{sinc}(f\tau)},\ |f| \le f_m$$

Attenuation relative to DC, worst at the top of the band $f = f_m$. The equaliser, placed with the reconstruction filter, undoes the magnitude distortion; the delay $\tau/2$ is simply left in. The effect is small when $\tau \ll T_s$: with $\tau/T_s = 0.1$ the loss even at $f_s/2$ is under 0.04 dB.

ZERO-ORDER HOLD (DAC) RECONSTRUCTION

$$H_0(f) = T_s\,\mathrm{sinc}(fT_s)\,e^{-j\pi fT_s}, \qquad \left|\frac{H_0(f_s/2)}{H_0(0)}\right| = \frac{2}{\pi} \approx 0.637 \ \ (-3.92\ \mathrm{dB})$$

Flat-top with $\tau = T_s$: each sample is held until the next, giving a staircase. It droops across the band (needs a $1/\mathrm{sinc}$ equaliser) and passes attenuated images (needs a smoothing lowpass filter).

MINIMUM BANDPASS SAMPLING RATE

$$f_{s,\min} = \frac{2f_H}{\lfloor f_H/B \rfloor}, \qquad B = f_H - f_L$$

$\lfloor \cdot \rfloor$ is the greatest-integer function. If f_H is an integer multiple of B, $f_{s,\min} = 2B$. In general $2B \le f_{s,\min} < 4B$.

ALL ALIAS-FREE UNIFORM SAMPLING RATES

$$\frac{2f_H}{k} \le f_s \le \frac{2f_L}{k-1}, \qquad k = 1, 2, \ldots, \left\lfloor \frac{f_H}{B} \right\rfloor$$

For $k = 1$ the upper limit is ∞: ordinary sampling at $f_s \ge 2f_H$. Rates in the gaps between these windows alias, so a rate above the minimum is not automatically safe.

CT TO DT FREQUENCY MAPPING

$$\Omega = \omega T_s = \frac{2\pi f}{f_s} \text{ rad/sample}, \qquad f = \frac{\Omega f_s}{2\pi}$$

One-to-one only on the baseband, $|\Omega| \le \pi \leftrightarrow |f| \le f_s/2$. $\cos(2\pi f_0 t)$ sampled gives $\cos(\Omega_0 n)$ with $\Omega_0 = 2\pi f_0/f_s$; reduce Ω_0 modulo 2π into $(-\pi, \pi]$ to read off the alias.

PERIODICITY OF A SAMPLED SINUSOID

$$x[n] = A\cos\left(2\pi\frac{f_0}{f_s}\,n + \theta\right) \text{ is periodic only if } \frac{f_0}{f_s} = \frac{p}{q} \text{ (integers, lowest terms, } q > 0); \text{ then } N = q$$

A continuous-time sinusoid is always periodic; its samples need not be: $\cos(n)$, from $f_0/f_s = 1/2\pi$, never repeats. Examples: 1 kHz sampled at 3 kHz gives $p/q = 1/3$, $N = 3$; 700 Hz sampled at 1 kHz gives $7/10$, $N = 10$. A sum of such sequences with periods N_1 and N_2 repeats every $\mathrm{lcm}(N_1, N_2)$ samples.

z-transform and discrete-time systems

BILATERAL Z-TRANSFORM

$$X(z) = \mathcal{Z}\{x[n]\} = \sum_{n=-\infty}^{\infty} x[n]\, z^{-n}$$

Defined only for z in the ROC, the set of z where the sum converges absolutely. Always quote $X(z)$ together with its ROC.

UNILATERAL Z-TRANSFORM

$$X^{+}(z) = \sum_{n=0}^{\infty} x[n]\, z^{-n}$$

Ignores $x[n]$ for $n < 0$; it equals the bilateral transform of $x[n]u[n]$. When it exists, its ROC is the exterior of a circle, $|z| > r$ (including $z = \infty$), so it is usually not written. For a causal sequence the two transforms are identical.

LINK TO THE DTFT

$$X(e^{j\Omega}) = X(z)\big|_{z=e^{j\Omega}}$$

Valid only when the ROC contains the unit circle $|z| = 1$. $X(e^{j\Omega})$ is periodic in Ω with period 2π.

RIGHT-SIDED EXPONENTIAL

$$a^n u[n] \;\leftrightarrow\; \frac{1}{1 - az^{-1}} = \frac{z}{z-a}, \qquad |z| > |a|$$

With $a = 1$: $u[n] \leftrightarrow \dfrac{z}{z-1}$, $|z| > 1$. With $a = 0$ (limit): $\delta[n] \leftrightarrow 1$ for all z.

LEFT-SIDED EXPONENTIAL

$$-a^n u[-n-1] \;\leftrightarrow\; \frac{z}{z-a}, \qquad |z| < |a|$$

Same algebraic transform as $a^n u[n]$; only the ROC differs. Note the minus sign and the $u[-n-1]$ (it starts at $n=-1$).

RAMP-WEIGHTED EXPONENTIAL

$$n\,a^n u[n] \;\leftrightarrow\; \frac{az^{-1}}{(1-az^{-1})^2} = \frac{az}{(z-a)^2}, \qquad |z| > |a|$$

Follows from differentiation in z. Useful variants: $nu[n] \leftrightarrow \frac{z}{(z-1)^2}$ and $(n+1)a^n u[n] \leftrightarrow \frac{z^2}{(z-a)^2}$, both right-sided.

DAMPED COSINE AND SINE

$$r^n \cos(\Omega_0 n)\, u[n] \;\leftrightarrow\; \frac{z(z - r\cos\Omega_0)}{z^2 - 2rz\cos\Omega_0 + r^2}$$

$$r^n \sin(\Omega_0 n)\, u[n] \;\leftrightarrow\; \frac{rz\sin\Omega_0}{z^2 - 2rz\cos\Omega_0 + r^2}$$

Here $r > 0$ and Ω_0 are real. ROC $|z| > r$; poles at $re^{\pm j\Omega_0}$. Put $r = 1$ for the undamped $\cos(\Omega_0 n)u[n]$ and $\sin(\Omega_0 n)u[n]$, whose poles lie on the unit circle (ROC $|z| > 1$).

TIME SHIFT (BILATERAL)

$$x[n-k] \;\leftrightarrow\; z^{-k}X(z)$$

ROC unchanged except possibly for $z = 0$ or $z = \infty$. A delay of k samples multiplies by z^{-k}. For the unilateral version with initial conditions see Section 5.

TIME REVERSAL

$$x[-n] \;\leftrightarrow\; X(1/z)$$

The ROC inverts: $r_1 < |z| < r_2$ becomes $1/r_2 < |z| < 1/r_1$. A pole at p moves to $1/p$.

SCALING IN Z (MULTIPLICATION BY AN EXPONENTIAL)

$$a^n x[n] \;\leftrightarrow\; X\!\left(\frac{z}{a}\right)$$

ROC scales to $|a|r_1 < |z| < |a|r_2$; every pole and zero moves from p to ap. With $a = e^{j\Omega_0}$ (modulation) the pole-zero plot rotates by Ω_0.

DIFFERENTIATION IN Z

$$n\,x[n] \;\leftrightarrow\; -z\,\frac{dX(z)}{dz}$$

Same ROC. Apply it twice for $n^2 x[n]$.

CONVOLUTION

$$x_1[n] * x_2[n] \;\leftrightarrow\; X_1(z)\,X_2(z)$$

ROC contains $R_1 \cap R_2$ (it can be larger if a pole cancels against a zero).

ACCUMULATION

$$\sum_{k=-\infty}^{n} x[k] \;\leftrightarrow\; \frac{X(z)}{1-z^{-1}} = \frac{z\,X(z)}{z-1}$$

ROC contains $R \cap \{|z| > 1\}$. The inverse operation, the first difference $x[n] - x[n-1]$, gives $(1-z^{-1})X(z)$.

CONJUGATION

$$x^*[n] \;\leftrightarrow\; X^*(z^*)$$

Same ROC. For a real sequence $X(z^*) = X^*(z)$, so complex poles and zeros come in conjugate pairs.

INITIAL VALUE THEOREM

$$x[0] = \lim_{z \to \infty} X(z)$$

For a causal $x[n]$ ($x[n] = 0$ for $n < 0$), or for any $x[n]$ with the unilateral transform. The next sample is $x[1] = \lim_{z \to \infty} z\,(X(z) - x[0])$. A rational $X(z)$ whose numerator degree in z exceeds the denominator degree cannot be the transform of a causal sequence, and the theorem does not apply. For an anticausal $x[n]$ ($x[n] = 0$ for $n > 0$) the mirror result is $x[0] = \lim_{z \to 0} X(z)$.

FINAL VALUE THEOREM

$$\lim_{n \to \infty} x[n] = \lim_{z \to 1} (z - 1)\, X(z) = \lim_{z \to 1} (1 - z^{-1})\, X(z)$$

For a causal $x[n]$ (or the unilateral transform of any $x[n]$), and only when the limit exists: every pole of $(z-1)X(z)$ must lie strictly inside the unit circle. $X(z)$ itself may have one simple pole at $z = 1$.

PARTIAL-FRACTION COEFFICIENT (SIMPLE POLE)

$$A_k = (z - p_k)\, \frac{X(z)}{z}\Big|_{z = p_k}$$

Equivalent form in z^{-1}: $X(z) = \sum_k \dfrac{A_k}{1 - p_k z^{-1}}$ with $A_k = (1 - p_k z^{-1}) X(z)\big|_{z = p_k}$. That form needs the numerator degree in z^{-1} to be less than the denominator degree; otherwise divide out the polynomial part first (it gives impulses).

INVERSION INTEGRAL AND RESIDUES

$$x[n] = \frac{1}{2\pi j} \oint_C X(z)\, z^{n-1}\, dz = \sum_{p_i \text{ inside } C} \operatorname{Res}\left[X(z)\, z^{n-1},\ p_i\right]$$

C is a counter-clockwise circle centred on the origin and lying in the ROC. For a causal $x[n]$ it encloses every pole. For small n, check whether z^{n-1} adds a pole at $z = 0$ (for example at $n = 0$ when $X(z)$ has no zero at the origin) and include its residue.

RESIDUE AT A POLE OF ORDER M

$$\operatorname{Res}_{z=p} = \frac{1}{(m-1)!} \lim_{z \to p} \frac{d^{m-1}}{dz^{m-1}} \left[(z - p)^m X(z)\, z^{n-1}\right]$$

For a simple pole ($m = 1$) this is just $(z - p)X(z)z^{n-1}$ evaluated at $z = p$.

SYSTEM FUNCTION OF A DIFFERENCE EQUATION

$$\sum_{k=0}^{N} a_k\, y[n-k] = \sum_{k=0}^{M} b_k\, x[n-k] \ \Rightarrow\ H(z) = \frac{\sum_{k=0}^{M} b_k z^{-k}}{\sum_{k=0}^{N} a_k z^{-k}}$$

Take the transform term by term with $x[n-k] \leftrightarrow z^{-k}X(z)$ and zero initial conditions. Sign convention: every output term is on the left. If the equation is written $y[n] = 0.5\, y[n-1] + x[n]$, the feedback coefficient changes sign when moved across: $H(z) = \dfrac{1}{1 - 0.5 z^{-1}}$.

FREQUENCY RESPONSE

$$H(e^{j\Omega}) = H(z)\big|_{z = e^{j\Omega}}$$

Exists when the ROC contains the unit circle (a stable system). $\Omega = 0$ is the point $z = 1$ (DC) and $\Omega = \pi$ is $z = -1$ (the highest digital frequency). $H(e^{j\Omega})$ has period 2π.

STEADY-STATE RESPONSE TO A SINUSOID

$$x[n] = A\cos(\Omega_0 n + \theta) \ \Rightarrow\ y[n] = A\left|H(e^{j\Omega_0})\right| \cos\left(\Omega_0 n + \theta + \angle H(e^{j\Omega_0})\right)$$

For a stable LTI system with real $h[n]$. It is exact for an input present for all n; for an input switched on at $n = 0$ it is the response once the transients have died out. A constant input A gives the output $A\,H(1)$.

UNILATERAL DELAY PROPERTY

$$x[n-k] \;\leftrightarrow\; z^{-k}X^{+}(z) + \sum_{m=1}^{k} x[-m]\, z^{-(k-m)}$$

The cases used most: $x[n-1] \leftrightarrow z^{-1}X^{+}(z) + x[-1]$ and $x[n-2] \leftrightarrow z^{-2}X^{+}(z) + z^{-1}x[-1] + x[-2]$.

UNILATERAL ADVANCE PROPERTY

$$x[n+k] \;\leftrightarrow\; z^{k}X^{+}(z) - \sum_{m=0}^{k-1} x[m]\, z^{k-m}$$

For $k=1$: $x[n+1] \leftrightarrow zX^{+}(z) - z\,x[0]$. For $k=2$: $z^2X^{+}(z) - z^2x[0] - z\,x[1]$.

THE S-TO-Z MAPPING

$$z = e^{sT}, \qquad s = \sigma + j\omega \;\Rightarrow\; |z| = e^{\sigma T},\;\; \angle z = \omega T$$

T is the sampling period. The digital frequency is $\Omega = \omega T = 2\pi f / f_s$ rad/sample. A continuous-time pole at $s = p$ appears at $z = e^{pT}$ in the transform of the sampled signal.

SAMPLED EXPONENTIAL

$$e^{-at}u(t) \;\to\; e^{-anT}u[n] \;\leftrightarrow\; \frac{z}{z - e^{-aT}}, \qquad |z| > e^{-aT}$$

For real a. The pole $s = -a$ maps to $z = e^{-aT}$, inside the unit circle when $a > 0$.

ZERO-ORDER HOLD WITH PLANT

$$G_h(s) = \frac{1 - e^{-sT}}{s}, \qquad G(z) = \mathcal{Z}\left\{\frac{1 - e^{-sT}}{s}\, G_p(s)\right\} = (1 - z^{-1})\,\mathcal{Z}\left\{\frac{G_p(s)}{s}\right\}$$

Exact at the sampling instants: $G(z)$ is $(1 - z^{-1})$ times the transform of the sampled step response of the plant.

ZOH WITH A FIRST-ORDER PLANT

$$G_p(s) = \frac{K}{s + a} \;\Rightarrow\; G(z) = \frac{K}{a} \cdot \frac{1 - e^{-aT}}{z - e^{-aT}}$$

For $a \neq 0$. From $\dfrac{K}{s(s+a)} = \dfrac{K}{a}\left(\dfrac{1}{s} - \dfrac{1}{s+a}\right)$. Without the hold, $\mathcal{Z}\{G_p(s)\} = \dfrac{Kz}{z - e^{-aT}}$, a different answer.

CASCADED BLOCKS IN A SAMPLED-DATA SYSTEM

$$\text{sampler between the blocks: } G_1(z)\,G_2(z), \qquad \text{no sampler between: } G_1G_2(z) = \mathcal{Z}\{G_1(s)\,G_2(s)\}$$

When a sampler separates $G_1(s)$ and $G_2(s)$, their pulse transfer functions multiply. When none does, multiply in s first and then transform; in general $G_1G_2(z) \neq G_1(z)\,G_2(z)$. Example with $G_1(s) = G_2(s) = \dfrac{1}{s+1}$ and $T = 1$: with a sampler between them, $\dfrac{z^2}{(z - e^{-1})^2}$; without, $\mathcal{Z}\left\{\dfrac{1}{(s+1)^2}\right\} = \dfrac{e^{-1}z}{(z - e^{-1})^2}$.

CLOSED LOOP WITH THE SAMPLER ON THE ERROR

$$\frac{C(z)}{R(z)} = \frac{G(z)}{1 + GH(z)}, \qquad GH(z) = \mathcal{Z}\{G(s)H(s)\}$$

$G(s)$ is the forward path after the sampler (hold included) and $H(s)$ the continuous feedback path. Because $G(s)$ and $H(s)$ are not separated by a sampler, they are multiplied before transforming. The loop is stable when every root of $1 + GH(z) = 0$ lies inside the unit circle.

STEADY-STATE ERROR OF A UNITY-FEEDBACK SAMPLED LOOP

$$e_{ss}\big|_{r=u(t)} = \frac{1}{1+\lim_{z\to 1} G(z)}, \qquad e_{ss}\big|_{r=t\,u(t)} = \frac{T}{\lim_{z\to 1}(z-1)\,G(z)}$$

Unity feedback, the sampler on the error, $G(z)$ the pulse transfer function of the forward path (hold included), and a stable closed loop (otherwise the final value theorem does not apply). From $E(z) = \dfrac{R(z)}{1+G(z)}$ and the final value theorem, with $R(z) = \dfrac{z}{z-1}$ for a step and $R(z) = \dfrac{Tz}{(z-1)^2}$ for the sampled ramp. The constants are often written $K_p = \lim_{z\to 1} G(z)$ and $K_v = \dfrac{1}{T}\lim_{z\to 1}(z-1)G(z)$, so that $e_{ss} = \dfrac{1}{1+K_p}$ for a step and $\dfrac{1}{K_v}$ for a ramp. Each pole of $G(z)$ at $z = 1$ plays the part of an integrator.

STABILITY TEST FOR A SECOND-ORDER CHARACTERISTIC POLYNOMIAL

$$P(z) = z^2 + a_1 z + a_0: \quad |a_0| < 1, \quad P(1) = 1 + a_1 + a_0 > 0, \quad P(-1) = 1 - a_1 + a_0 > 0$$

For real coefficients, all three conditions together are necessary and sufficient for both roots to lie strictly inside the unit circle (the Jury conditions for order 2). For $a_2 z^2 + a_1 z + a_0$ with $a_2 > 0$ the first condition becomes $|a_0| < a_2$. At any order n, with a positive leading coefficient, $P(1) > 0$ and $(-1)^n P(-1) > 0$ are necessary. For a full test at higher orders, use the Jury table, or substitute $z = \dfrac{1+w}{1-w}$ and apply the Routh test to the polynomial in w.

DTFT, DFT and FFT

DTFT (ANALYSIS EQUATION)

$$X(e^{j\omega}) = \sum_{n=-\infty}^{\infty} x[n]\, e^{-j\omega n}$$

Bilateral sum over all n, with ω in rad/sample. Defined when the sum converges (see existence below).

INVERSE DTFT (SYNTHESIS EQUATION)

$$x[n] = \frac{1}{2\pi}\int_{-\pi}^{\pi} X(e^{j\omega})\, e^{j\omega n}\, d\omega$$

The integral may be taken over any interval of length 2π. In the ω convention the $1/2\pi$ belongs to the inverse.

VALUES AT SPECIAL POINTS

$$X(e^{j0}) = \sum_n x[n], \qquad X(e^{j\pi}) = \sum_n (-1)^n x[n], \qquad x[0] = \frac{1}{2\pi}\int_{-\pi}^{\pi} X(e^{j\omega})\, d\omega$$

Put $\omega = 0$ or $\omega = \pi$ in the analysis equation, or $n = 0$ in the inverse. These evaluate such sums and integrals without finding $X(e^{j\omega})$ at all.

RIGHT-SIDED EXPONENTIAL

$$a^n u[n] \;\leftrightarrow\; \frac{1}{1-ae^{-j\omega}}, \qquad |a| < 1$$

For real a the magnitude is $1/\sqrt{1-2a\cos\omega+a^2}$. With $0 < a < 1$ it peaks at $1/(1-a)$ at $\omega = 0$ and falls to $1/(1+a)$ at $\omega = \pi$ (low-pass); for $-1 < a < 0$ the roles swap (high-pass). For $|a| \geq 1$ there is no DTFT as an ordinary function.

DIRICHLET KERNEL (LENGTH-N RECTANGULAR PULSE)

$$\sum_{n=0}^{N-1} e^{-j\omega n} = e^{-j\omega(N-1)/2}\,\frac{\sin(N\omega/2)}{\sin(\omega/2)}$$

Value N at $\omega = 0$ (as a limit). Zeros at $\omega = 2\pi m/N$ for integers m that are not multiples of N. The main lobe is $4\pi/N$ rad/sample wide, null to null. Unlike the continuous-time sinc, the whole expression is periodic with period 2π. The ratio $\sin(N\omega/2)/\sin(\omega/2)$ on its own has period 2π only for odd N; for even N it changes sign after 2π, and the linear-phase factor changes sign too, so the product is still 2π-periodic.

IDEAL LOW-PASS (SINC) SEQUENCE

$$h[n] = \frac{\sin(\omega_c n)}{\pi n} = \frac{\omega_c}{\pi}\,\mathrm{sinc}\left(\frac{\omega_c n}{\pi}\right) \;\leftrightarrow\; H(e^{j\omega}) = 1 \text{ for } |\omega| \le \omega_c,\;\; 0 \text{ for } \omega_c < |\omega| \le \pi$$

$h[0] = \omega_c/\pi$ (as a limit). The sequence is not absolutely summable but is square summable, with energy $\sum_n h^2[n] = \omega_c/\pi$ by Parseval. It is non-causal and infinitely long, so it cannot be realised exactly.

TIME SHIFT AND FREQUENCY SHIFT

$$x[n-n_0] \leftrightarrow e^{-j\omega n_0} X(e^{j\omega}), \qquad e^{j\omega_0 n} x[n] \leftrightarrow X(e^{j(\omega-\omega_0)})$$

A delay of n_0 samples adds the linear phase $-\omega n_0$ and leaves $|X|$ unchanged. Multiplying by $(-1)^n = e^{j\pi n}$ shifts the spectrum by π, which turns a low-pass sequence into a high-pass one.

SYMMETRY OF A REAL SEQUENCE

$$x[n] \text{ real} \;\Rightarrow\; X(e^{-j\omega}) = X^*(e^{j\omega})$$

So $|X|$ and $\mathrm{Re}\,X$ are even in ω, while $\angle X$ and $\mathrm{Im}\,X$ are odd. A real, even $x[n]$ has a real, even X; a real, odd $x[n]$ has a purely imaginary, odd X. More generally, for real $x[n]$ the even part $\frac{1}{2}(x[n]+x[-n])$ transforms to $\mathrm{Re}\,X(e^{j\omega})$ and the odd part $\frac{1}{2}(x[n]-x[-n])$ to $j\,\mathrm{Im}\,X(e^{j\omega})$.

CONVOLUTION THEOREM

$$y[n] = x[n] * h[n] = \sum_{k=-\infty}^{\infty} x[k]\,h[n-k] \;\leftrightarrow\; Y(e^{j\omega}) = X(e^{j\omega})\,H(e^{j\omega})$$

Linear (aperiodic) convolution. This is why the output spectrum of an LTI system is the input spectrum times its frequency response.

MULTIPLICATION (PERIODIC CONVOLUTION)

$$x[n]\,y[n] \leftrightarrow \frac{1}{2\pi}\int_{-\pi}^{\pi} X(e^{j\theta})\,Y(e^{j(\omega-\theta)})\,d\theta$$

The integral runs over one period only, so this is a periodic convolution. Windowing $x[n]$ by $w[n]$ smears X with the window's kernel, which is the origin of spectral leakage.

DIFFERENTIATION IN FREQUENCY

$$n\,x[n] \leftrightarrow j\,\frac{dX(e^{j\omega})}{d\omega}, \qquad \sum_n n\,x[n] = j\,\frac{dX(e^{j\omega})}{d\omega}\bigg|_{\omega=0}$$

Useful for $\sum n\,x[n]$ and, together with Parseval, for $\int_{-\pi}^{\pi} |dX/d\omega|^2\,d\omega = 2\pi\sum n^2|x[n]|^2$.

PARSEVAL'S THEOREM

$$\sum_{n=-\infty}^{\infty} |x[n]|^2 = \frac{1}{2\pi}\int_{-\pi}^{\pi} |X(e^{j\omega})|^2\,d\omega$$

Energy in time equals the energy spread over one period of frequency. The general form is $\sum_n x[n]\,y^*[n] = \frac{1}{2\pi}\int_{-\pi}^{\pi} X(e^{j\omega})\,Y^*(e^{j\omega})\,d\omega$.

SINUSOIDAL RESPONSE OF A STABLE LTI SYSTEM

$$x[n] = A\cos(\omega_0 n + \phi) \;\Rightarrow\; y[n] = A\,|H(e^{j\omega_0})|\cos\left(\omega_0 n + \phi + \angle H(e^{j\omega_0})\right)$$

For real $h[n]$ and an input applied for all n. Evaluate H at the input frequency in rad/sample; reduce ω_0 modulo 2π first if it is outside $(-\pi, \pi]$.

DFT

$$X[k] = \sum_{n=0}^{N-1} x[n]\,W_N^{kn} = \sum_{n=0}^{N-1} x[n]\,e^{-j2\pi kn/N}, \qquad k = 0, 1, \ldots, N-1$$

$W_N = e^{-j2\pi/N}$. $X[k]$ is complex in general, even when $x[n]$ is real.

INVERSE DFT (IDFT)

$$x[n] = \frac{1}{N}\sum_{k=0}^{N-1} X[k]\, W_N^{-kn} = \frac{1}{N}\sum_{k=0}^{N-1} X[k]\, e^{j2\pi kn/N}, \qquad n = 0, 1, \ldots, N-1$$

Some texts put $1/\sqrt{N}$ in both directions (the unitary DFT), and then Parseval has no $1/N$. These notes keep $1/N$ in the inverse only.

DFT AS SAMPLES OF THE DTFT

$$X[k] = X(e^{j\omega})\big|_{\omega = 2\pi k/N}, \qquad k = 0, 1, \ldots, N-1$$

Exact when $x[n]$ is zero outside $0 \le n \le N-1$; a length-L sequence with $L \le N$ is zero-padded to N. The samples are spaced $2\pi/N$ apart over one period $[0, 2\pi)$.

SAMPLING THE DTFT ALIASES IN TIME

$$\mathrm{IDFT}_N\left\{X(e^{j2\pi k/N})\right\} = \sum_{r=-\infty}^{\infty} x[n - rN], \qquad 0 \le n \le N-1$$

If $x[n]$ is longer than N (or infinitely long), N samples of its DTFT recover this time-aliased sum, not $x[n]$. It is the time-domain counterpart of the aliasing caused by sampling a signal too slowly.

DTFT OF A PERIODIC SEQUENCE (DISCRETE FOURIER SERIES)

$$\tilde{x}[n+N] = \tilde{x}[n] \;\Rightarrow\; \tilde{x}[n] = \frac{1}{N}\sum_{k=0}^{N-1} \tilde{X}[k]\, e^{j2\pi kn/N} \;\leftrightarrow\; \frac{2\pi}{N}\sum_{k=-\infty}^{\infty} \tilde{X}[k]\, \delta\left(\omega - \frac{2\pi k}{N}\right)$$

$\tilde{X}[k] = \sum_{n=0}^{N-1} \tilde{x}[n]\, W_N^{kn}$ is the N-point DFT of one period, extended periodically in k. A periodic sequence is not summable, so its DTFT is a train of impulses at the bin frequencies $2\pi k/N$. If the Fourier-series coefficients are defined as $a_k = \tilde{X}[k]/N$, the same result reads $2\pi \sum_k a_k\, \delta(\omega - 2\pi k/N)$. Check: $\tilde{x}[n] = 1$ gives $\tilde{X}[k] = N$ at multiples of N and 0 elsewhere, hence $2\pi \sum_k \delta(\omega - 2\pi k)$.

FREQUENCY OF BIN K

$$\omega_k = \frac{2\pi k}{N}\ \text{rad/sample}, \qquad F_k = \frac{kF_s}{N}\ \text{Hz}$$

For a sequence sampled at F_s. Bins k and $N-k$ correspond to $\pm F_k$: for $k > N/2$, bin k represents the negative frequency $(k-N)F_s/N$. For even N, bin $N/2$ is $F_s/2$.

DFT MATRICES FOR N = 2 AND N = 4

$$F_2 = \begin{bmatrix} 1 & 1 \\ 1 & -1 \end{bmatrix}, \qquad F_4 = \begin{bmatrix} 1 & 1 & 1 & 1 \\ 1 & -j & -1 & j \\ 1 & -1 & 1 & -1 \\ 1 & j & -1 & -j \end{bmatrix}$$

$\mathbf{X} = F_N \mathbf{x}$ with $[F_N]_{kn} = W_N^{kn}$ (row k, column n, both counted from 0). F_N is symmetric. With F_4 memorised, a 4-point DFT needs only additions, subtractions and multiplications by $\pm j$ (a swap of real and imaginary parts with a sign change).

INVERSE AND POWERS OF THE DFT MATRIX

$$F_N^{-1} = \frac{1}{N}F_N^{*}, \qquad F_N^2\,\mathbf{x} = N\,[x[0],\ x[N-1],\ x[N-2],\ \ldots,\ x[1]]^T, \qquad F_N^4 = N^2 I$$

$F_N F_N^{*} = NI$, so $F_N/\sqrt{N}$ is unitary. Applying the DFT twice returns the circularly time-reversed sequence $x[(-n)_N]$ scaled by N; four applications return $N^2\mathbf{x}$. Hence every eigenvalue of F_N is one of $\pm\sqrt{N}$, $\pm j\sqrt{N}$.

CONJUGATE SYMMETRY FOR A REAL SEQUENCE

$$x[n] \text{ real } \Rightarrow\ X[N-k] = X^*[k], \quad |X[N-k]| = |X[k]|, \quad \angle X[N-k] = -\angle X[k]$$

For $1 \le k \le N-1$. So $X[0]$ is real and, for even N, $X[N/2]$ is real. Only $X[0]$ to $X[N/2]$ need computing for even N ($X[0]$ to $X[(N-1)/2]$ for odd N); the rest are conjugates. If $x[n]$ is real and circularly even, $x[n] = x[(N-n)_N]$, then $X[k]$ is real and circularly even; if it is real and circularly odd, $x[n] = -x[(N-n)_N]$, then $X[k]$ is purely imaginary and circularly odd.

CIRCULAR CONVOLUTION

$$y[n] = x[n] \circledast h[n] = \sum_{m=0}^{N-1} x[m]\, h[(n-m)_N], \quad 0 \le n \le N-1 \ \leftrightarrow\ Y[k] = X[k]\, H[k]$$

Both sequences are treated as length N (zero-padded if shorter). Multiplying two N-point DFTs and inverting always gives the circular convolution, not the linear one.

PARSEVAL'S RELATION FOR THE DFT

$$\sum_{n=0}^{N-1} |x[n]|^2 = \frac{1}{N} \sum_{k=0}^{N-1} |X[k]|^2, \qquad \sum_{n=0}^{N-1} x[n]\, y^*[n] = \frac{1}{N} \sum_{k=0}^{N-1} X[k]\, Y^*[k]$$

With $W_N = e^{-j2\pi/N}$ and $1/N$ in the inverse, the $1/N$ is on the frequency side. Leaving it out gives an answer N times too large.

VALUES AT K = 0, N = 0 AND N/2

$$X[0] = \sum_{n=0}^{N-1} x[n], \qquad x[0] = \frac{1}{N} \sum_{k=0}^{N-1} X[k], \qquad X[N/2] = \sum_{n=0}^{N-1} (-1)^n x[n], \qquad x[N/2] = \frac{1}{N} \sum_{k=0}^{N-1} (-1)^k X[k]$$

The last two need N even. $X[0]$ is N times the mean of $x[n]$, and $x[0]$ is the mean of the $X[k]$.

CIRCULAR CONVOLUTION IS ALIASED LINEAR CONVOLUTION

$$y_c[n] = \sum_{r=-\infty}^{\infty} y_l[n - rN], \qquad 0 \le n \le N-1$$

Valid whenever $N \ge \max(N_1, N_2)$. For $\max(N_1, N_2) \le N < N_1 + N_2 - 1$, only the first $N_1 + N_2 - 1 - N$ samples can differ from the linear result; samples $n = N_1 + N_2 - 1 - N$ to $N-1$ equal it.

LENGTH NEEDED FOR LINEAR CONVOLUTION BY DFT

$$N \ge N_1 + N_2 - 1$$

N_1 and N_2 are the lengths of the two sequences. For a radix-2 FFT, take the next power of 2 at or above $N_1 + N_2 - 1$.

BIN SPACING

$$\Delta\omega = \frac{2\pi}{N} \text{ rad/sample}, \qquad \Delta F = \frac{F_s}{N} \text{ Hz}$$

N is the DFT length, including any zero padding.

RECORD-LENGTH RESOLUTION (RECTANGULAR WINDOW)

$$\Delta F_{\text{res}} \approx \frac{F_s}{L} = \frac{1}{L\,T_s} = \frac{1}{T_0}$$

L is the number of genuine data samples and $T_0 = LT_s$ the observation time. This is a rule of thumb, not an exact threshold: the main lobe of the rectangular window is $2F_s/L$ wide null to null, and whether two tones about F_s/L apart show as two peaks also depends on their relative phase. Only a longer record improves it. Other windows lower the side lobes at the cost of a wider main lobe.

DECIMATION-IN-TIME (DIT) SPLIT

$$X[k] = X_e[k] + W_N^k\, X_o[k], \qquad X[k+N/2] = X_e[k] - W_N^k\, X_o[k], \qquad k = 0, 1, \ldots, N/2-1$$

$X_e[k]$ and $X_o[k]$ are the $N/2$-point DFTs of the even-indexed samples $x[2r]$ and the odd-indexed samples $x[2r+1]$. Repeat the split until only 2-point DFTs remain.

DECIMATION-IN-FREQUENCY (DIF) SPLIT

$$X[2r] = \sum_{n=0}^{N/2-1} \left(x[n] + x[n+N/2]\right) W_{N/2}^{rn}, \qquad X[2r+1] = \sum_{n=0}^{N/2-1} \left(x[n] - x[n+N/2]\right) W_N^n\, W_{N/2}^{rn}$$

For $r = 0, 1, \ldots, N/2-1$. The first and second halves of the input are combined, and the even- and odd-indexed outputs each come from an $N/2$-point DFT.

COST OF A DIRECT DFT

$$M_{\text{DFT}} = N^2, \qquad A_{\text{DFT}} = N(N-1)$$

Complex multiplications M and complex additions A: each of the N outputs needs N products and $N-1$ additions.

COST OF A RADIX-2 FFT

$$M_{\text{FFT}} = \frac{N}{2}\log_2 N, \qquad A_{\text{FFT}} = N\log_2 N$$

$\log_2 N$ stages of $N/2$ butterflies. The count includes trivial multiplications by $W_N^0 = 1$ and by $-j$, which a practical program skips; use the formula as stated unless a question says otherwise. One complex multiplication, done directly as $(a+jb)(c+jd) = (ac-bd) + j(ad+bc)$, costs 4 real multiplications and 2 real additions.

FIR and IIR filters

FIR FILTER OF LENGTH N

$$y[n] = \sum_{k=0}^{N-1} b_k\, x[n-k], \qquad h[n] = b_n\ (0 \le n \le N-1), \qquad H(z) = \sum_{k=0}^{N-1} h[k]\, z^{-k}$$

Order $N-1$, taking $h[0] \neq 0$ and $h[N-1] \neq 0$. Writing $H(z) = \left(h[0]z^{N-1} + h[1]z^{N-2} + \cdots + h[N-1]\right)/z^{N-1}$ shows that its only poles are $N-1$ poles at $z = 0$. So a causal FIR filter is always BIBO stable.

IIR FILTER AND ITS SIGN CONVENTION

$$y[n] = \sum_{k=0}^{M} b_k\, x[n-k] - \sum_{k=1}^{K} a_k\, y[n-k] \iff H(z) = \frac{\sum_{k=0}^{M} b_k z^{-k}}{1 + \sum_{k=1}^{K} a_k z^{-k}}$$

Numerator order M, denominator order K. The minus sign in the difference equation puts $+a_k$ in the denominator. So an equation written as $y[n] = a\,y[n-1] + x[n]$ gives $H(z) = 1/(1-az^{-1})$, with a pole at $z = a$. The causal filter is stable if and only if every pole left after cancelling common factors satisfies $|p_k| < 1$.

LINEAR-PHASE CONDITION AND FREQUENCY RESPONSE

$$h[n] = h[N-1-n] \;\Rightarrow\; H(e^{j\omega}) = A(\omega)\, e^{-j\omega(N-1)/2}$$

$$h[n] = -h[N-1-n] \;\Rightarrow\; H(e^{j\omega}) = j\,A(\omega)\, e^{-j\omega(N-1)/2} = A(\omega)\, e^{j\left(\frac{\pi}{2} - \omega\frac{N-1}{2}\right)}$$

$A(\omega)$ is real. Antisymmetry forces the middle sample of an odd-length filter to be zero, since $h[\alpha] = -h[\alpha]$.

GROUP DELAY OF EVERY LINEAR-PHASE FIR FILTER

$$\tau_g = -\frac{d\theta(\omega)}{d\omega} = \frac{N-1}{2} \text{ samples}$$

The same for all four types, at every ω where $A(\omega) \neq 0$. It is an integer for odd N (Types I and III) and a half-integer for even N (Types II and IV). In seconds it is $\tau_g T$.

SYMMETRY IN THE Z-DOMAIN AND THE FORCED ZEROS

$$H(z) = \pm\, z^{-(N-1)}\, H(z^{-1}) \qquad (+\ \text{symmetric},\ -\ \text{antisymmetric})$$

Put $z = \pm 1$, where $z^{-1} = z$. Symmetric with N even: $H(-1) = (-1)^{N-1}H(-1) = -H(-1)$, so $H(-1) = 0$. Antisymmetric: $H(1) = -H(1)$, so $H(1) = 0$ always. With N odd, also $H(-1) = -H(-1)$, so $H(-1) = 0$.

DELAYED IDEAL LOWPASS IMPULSE RESPONSE

$$h_d[n] = \frac{\sin\left(\omega_c(n-\alpha)\right)}{\pi(n-\alpha)}, \quad n \neq \alpha; \qquad h_d[\alpha] = \frac{\omega_c}{\pi}; \qquad \alpha = \frac{N-1}{2}$$

ω_c is in rad/sample. It is written with $\sin$ explicitly, so no sinc convention is needed. The centre value is the limit as $n \to \alpha$. For even N, α is a half-integer and no sample falls at $n = \alpha$.

OTHER IDEAL RESPONSES BUILT FROM THE LOWPASS

$$h_{d,\mathrm{HP}}[n] = \delta[n-\alpha] - h_{d,\mathrm{LP}}[n] \quad (\text{same } \omega_c)$$
$$h_{d,\mathrm{BP}}[n] = h_{d,\mathrm{LP}}[n]\big|_{\omega_c=\omega_2} - h_{d,\mathrm{LP}}[n]\big|_{\omega_c=\omega_1} \quad (\omega_1 < \omega_2)$$
$$h_{d,\mathrm{BS}}[n] = \delta[n-\alpha] - h_{d,\mathrm{BP}}[n]$$

The $\delta[n-\alpha]$ term needs an integer α, i.e. odd N. So a windowed highpass or bandstop design is Type I, consistent with the forced zero at $z = -1$ that rules out Type II.

COMMON WINDOWS (0 TO N-1, ZERO ELSEWHERE)

$$\text{Rectangular: } w[n] = 1$$
$$\text{Hann: } w[n] = 0.5 - 0.5\cos\frac{2\pi n}{N-1}$$
$$\text{Hamming: } w[n] = 0.54 - 0.46\cos\frac{2\pi n}{N-1}$$
$$\text{Blackman: } w[n] = 0.42 - 0.5\cos\frac{2\pi n}{N-1} + 0.08\cos\frac{4\pi n}{N-1}$$

For $0 \le n \le N-1$. This is one common form; some texts use $N+1$ in place of $N-1$ so that the end samples of the Hann window are not zero. All four are symmetric about $n = (N-1)/2$, so linear phase is kept. The Hamming window's end samples are 0.08, not zero.

FREQUENCY-SAMPLING DESIGN

$$H[k] = H_d\left(e^{j2\pi k/N}\right), \quad k = 0, \ldots, N-1; \qquad h[n] = \frac{1}{N}\sum_{k=0}^{N-1} H[k]\, e^{j2\pi kn/N}, \quad 0 \le n \le N-1$$

This is the N-point inverse DFT, with $W_N = e^{-j2\pi/N}$ so that $h[n] = \frac{1}{N}\sum_k H[k]W_N^{-kn}$. The designed response passes exactly through the N specified samples and is interpolated between them. For a real, linear-phase Type I result (N odd), take $H[k] = |H[k]|\, e^{-j2\pi k\alpha/N}$ with $|H[N-k]| = |H[k]|$.

IMPULSE INVARIANCE

$$h[n] = T\, h_a(nT); \qquad H_a(s) = \sum_{k=1}^{K}\frac{A_k}{s-p_k} \implies H(z) = \sum_{k=1}^{K}\frac{TA_k}{1-e^{p_kT}z^{-1}}$$

Needs a strictly proper $H_a(s)$ with simple poles (do the partial fractions first). The sample at $n = 0$ uses $h_a(0^+)$. Each analog pole p_k becomes the digital pole e^{p_kT}; zeros do **not** map by this rule. With the factor T, $H(e^{j\omega}) \approx H_a(j\omega/T)$ for $|\omega| \le \pi$ when aliasing is small. Some texts use $h[n] = h_a(nT)$ without it, which scales $H(z)$ by $1/T$, so check which one a question uses.

BILINEAR TRANSFORMATION

$$s = \frac{2}{T}\cdot\frac{1-z^{-1}}{1+z^{-1}} = \frac{2}{T}\cdot\frac{z-1}{z+1}, \qquad z = \frac{1+sT/2}{1-sT/2}$$

Substitute into $H_a(s)$ to get an $H(z)$ of the same order. The whole left half s-plane maps inside $|z|=1$, so stability is preserved. The entire $j\Omega$ axis maps once onto the unit circle: $\Omega = 0$ goes to $\omega = 0$ and $\Omega \to \pm\infty$ goes to $\omega = \pm\pi$. There is no aliasing.

FREQUENCY WARPING OF THE BILINEAR TRANSFORMATION

$$\Omega = \frac{2}{T}\tan\frac{\omega}{2} \qquad \Longleftrightarrow \qquad \omega = 2\tan^{-1}\frac{\Omega T}{2}$$

Ω in rad/s (analog), ω in rad/sample (digital). The relation is nearly linear, $\omega \approx \Omega T$, only for small ω. The whole infinite analog axis is squeezed into $0 \le \omega < \pi$.

BUTTERWORTH MAGNITUDE RESPONSE

$$|H_a(j\Omega)|^2 = \frac{1}{1+\left(\Omega/\Omega_c\right)^{2N}}$$

Order N (in this section N is the prototype's order, not an FIR length), cut-off Ω_c in rad/s. $|H_a(j0)| = 1$ and $|H_a(j\Omega_c)| = 1/\sqrt{2}$, i.e. -3.01 dB at the cut-off for every N. The response is monotonic in both bands. It is maximally flat: the first $2N-1$ derivatives of $|H_a|^2$ are zero at $\Omega = 0$. The asymptotic roll-off is $-20N$ dB/decade.

BUTTERWORTH ORDER

$$N \ge \frac{\log_{10}\left[\dfrac{10^{0.1A_s}-1}{10^{0.1A_p}-1}\right]}{2\log_{10}\left(\Omega_s/\Omega_p\right)}$$

A_p is the maximum attenuation (dB) allowed over the passband $0 \le \Omega \le \Omega_p$. Because the response is monotonic, the attenuation there is largest at Ω_p. A_s is the minimum stopband attenuation (dB) at Ω_s. Round **up** to the next integer. If the passband edge is the 3-dB cut-off ($A_p = 10\log_{10}2 \approx 3.01$ dB, $\Omega_p = \Omega_c$), then $10^{0.1A_p} - 1 = 1$.

BUTTERWORTH POLES

$$s_k = \Omega_c\, e^{\,j\pi(2k+N-1)/(2N)}, \qquad k = 1, 2, \ldots, N; \qquad H_a(s) = \frac{\Omega_c^N}{\prod_{k=1}^{N}(s - s_k)}$$

The poles are equally spaced by π/N on a circle of radius Ω_c, all in the left half-plane and symmetric about the real axis. There is a real pole at $-\Omega_c$ only when N is odd.

NORMALISED BUTTERWORTH DENOMINATORS (UNIT CUT-OFF)

$$N = 1:\ s + 1$$
$$N = 2:\ s^2 + \sqrt{2}\,s + 1$$
$$N = 3:\ (s+1)(s^2 + s + 1)$$
$$N = 4:\ (s^2 + 0.7654\,s + 1)(s^2 + 1.8478\,s + 1)$$

For a cut-off Ω_c, replace s by s/Ω_c. Each quadratic factor is $s^2 + 2\sin\left(\frac{(2k-1)\pi}{2N}\right)s + 1$ for $k = 1, \ldots, \lfloor N/2 \rfloor$, with a factor $s+1$ added when N is odd. For $N = 2$ the factor $s^2 + \sqrt{2}\,s + 1$ has damping ratio $\zeta = 1/\sqrt{2}$.

CHEBYSHEV TYPE I (IN BRIEF)

$$|H_a(j\Omega)|^2 = \frac{1}{1+\varepsilon^2\,T_N^2(\Omega/\Omega_p)}$$
$$T_N(x) = \cos(N\cos^{-1}x)\ \ (|x| \le 1), \qquad T_N(x) = \cosh(N\cosh^{-1}x)\ \ (x > 1)$$

Equiripple in the passband, monotonic in the stopband. Passband ripple $= 10\log_{10}(1+\varepsilon^2)$ dB. At DC, $|H_a| = 1$ for odd N and $1/\sqrt{1+\varepsilon^2}$ for even N. T_N obeys $T_{N+1}(x) = 2xT_N(x) - T_{N-1}(x)$ with $T_0 = 1$, $T_1 = x$.

ANALOG FREQUENCY TRANSFORMATIONS (FROM A LOWPASS PROTOTYPE WITH CUT-OFF 1 RAD/S)

$$\text{LP, cut-off } \Omega_c: \quad s \to \frac{s}{\Omega_c}$$

$$\text{HP, cut-off } \Omega_c: \quad s \to \frac{\Omega_c}{s}$$

$$\text{BP, edges } \Omega_l, \Omega_u: \quad s \to \frac{s^2 + \Omega_0^2}{s\,B}$$

$$\text{BS, edges } \Omega_l, \Omega_u: \quad s \to \frac{s\,B}{s^2 + \Omega_0^2}$$

Here $\Omega_0^2 = \Omega_l \Omega_u$ and $B = \Omega_u - \Omega_l$. The prototype's edge at ± 1 rad/s lands on Ω_c, or on Ω_l and Ω_u. The bandpass and bandstop substitutions double the order. In a bilinear design, prewarp every digital edge first, then apply the substitution to the prototype.

GAIN FROM THE POLE-ZERO PLOT

$$H(z) = b_0 \frac{\prod_k (1 - z_k z^{-1})}{\prod_k (1 - p_k z^{-1})} \;\Rightarrow\; |H(e^{j\omega})| = |b_0| \frac{\prod_k |e^{j\omega} - z_k|}{\prod_k |e^{j\omega} - p_k|}$$

This uses $|1 - z_k e^{-j\omega}| = |e^{j\omega} - z_k|$. Poles or zeros at the origin are at distance 1 from every point on the circle, so they change only the phase.

SECOND-ORDER NOTCH FILTER

$$H(z) = \frac{1 - 2\cos\omega_0\, z^{-1} + z^{-2}}{1 - 2r\cos\omega_0\, z^{-1} + r^2 z^{-2}}, \qquad 0 < r < 1$$

Zeros at $e^{\pm j\omega_0}$ give an exact null at ω_0; poles at $r e^{\pm j\omega_0}$. The closer r is to 1, the narrower the notch. With r close to 1, the gain away from ω_0 is close to 1. Without the poles, $1 - 2\cos\omega_0\, z^{-1} + z^{-2}$ is an FIR notch with a much wider dip.

ALL-PASS FILTER

$$H_{ap}(z) = \frac{z^{-1} - a^*}{1 - a z^{-1}}, \quad |a| < 1 \;\Rightarrow\; |H_{ap}(e^{j\omega})| = 1 \text{ for all } \omega$$

Pole at $z = a$, zero at $1/a^*$. Higher orders are products of such sections. With a real denominator $A(z)$ of order K, $H_{ap}(z) = z^{-K} A(z^{-1})/A(z)$, i.e. the numerator coefficients are the denominator coefficients in reverse order. Used to correct phase (delay) without changing the magnitude.

L-POINT MOVING AVERAGE

$$H(z) = \frac{1}{L}\sum_{k=0}^{L-1} z^{-k} = \frac{1}{L}\cdot\frac{1 - z^{-L}}{1 - z^{-1}}, \qquad |H(e^{j\omega})| = \left|\frac{\sin(\omega L/2)}{L\sin(\omega/2)}\right|$$

A lowpass, symmetric (Type I or II) FIR filter with DC gain 1 and group delay $(L-1)/2$. Its zeros are at $e^{j2\pi k/L}$, $k = 1, \ldots, L-1$ (the $k = 0$ zero cancels the pole at $z = 1$), so it has nulls at $\omega = 2\pi k/L$.

COMB FILTERS

$$H(z) = 1 - z^{-L}: \quad |H(e^{j\omega})| = 2\left|\sin\frac{\omega L}{2}\right|, \text{ nulls at } \omega = \frac{2\pi k}{L}$$

$$H(z) = 1 + z^{-L}: \quad |H(e^{j\omega})| = 2\left|\cos\frac{\omega L}{2}\right|, \text{ nulls at } \omega = \frac{(2k+1)\pi}{L}$$

k is an integer. $1 - z^{-L}$ removes DC and every multiple of $2\pi/L$, for example a mains frequency and its harmonics when f_s/L equals the mains frequency. Both are linear-phase FIR filters (antisymmetric and symmetric respectively) with group delay $L/2$.

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