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GATE DA

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Linear Algebra Formula Sheet

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Vector spaces, subspaces, and linear dependence and independence

SUBSPACE TEST

$$W \subseteq V \text{ is a subspace } \iff \mathbf{0} \in W \text{ and } \mathbf{u}, \mathbf{v} \in W,\ c \in \mathbb{R} \Rightarrow \mathbf{u} + \mathbf{v} \in W \text{ and } c\mathbf{u} \in W$$

Applies to any subset W of a vector space V. If the zero vector is missing, W fails immediately.

SUBSPACES UNDER INTERSECTION AND SUM

$$\dim(W_1 + W_2) = \dim W_1 + \dim W_2 - \dim(W_1 \cap W_2)$$

W1 and W2 are finite-dimensional subspaces of a vector space V. The intersection and the sum W1+W2 (all u+v with u in W1, v in W2) are always subspaces. The union is a subspace exactly when one of the two contains the other.

SPAN

$$\operatorname{span}\{\mathbf{v}_1, \ldots, \mathbf{v}_k\} = \{c_1\mathbf{v}_1 + \cdots + c_k\mathbf{v}_k : c_i \in \mathbb{R}\}$$

The span is always a subspace, and it is the smallest subspace containing the given vectors.

MEMBERSHIP IN A COLUMN SPACE

$$\mathbf{b} \in \operatorname{span}\{\text{columns of } A\} \iff A\mathbf{x} = \mathbf{b} \text{ is consistent} \iff \operatorname{rank}[A \mid \mathbf{b}] = \operatorname{rank} A$$

A is m x n real, b is in R^m. Use it to decide whether a vector lies in the span of given vectors.

DEFINITION OF LINEAR INDEPENDENCE

$$c_1\mathbf{v}_1 + \cdots + c_k\mathbf{v}_k = \mathbf{0} \Rightarrow c_1 = \cdots = c_k = 0$$

The vectors are independent if only the trivial combination gives the zero vector; otherwise they are dependent. For k >= 2 this is equivalent to: no vector in the set is a combination of the others. Also equivalent: the list v1, ..., vk is dependent exactly when v1 = 0 or some vj is a combination of v1, ..., v(j-1).

BASIS FACTS

$$\dim V = n \Rightarrow \text{any } n \text{ independent vectors, or any } n \text{ spanning vectors, form a basis}$$

V is finite-dimensional. Any independent set has at most n vectors, any spanning set has at least n, and an independent set can be extended to a basis. If W is a subspace of V, then dim W <= n, with equality only when W = V.

DIMENSIONS OF THE MATRIX SUBSPACES

$$\dim \operatorname{Col}(A) = \dim \operatorname{Row}(A) = r, \qquad \dim \operatorname{Null}(A) = n - r, \qquad \dim \operatorname{Null}(A^T) = m - r$$

A is real m x n of rank r. Row rank equals column rank, so rank(A) = rank(A^T), and r <= min(m, n). The middle result, rank plus nullity equals the number of columns n, is the rank-nullity theorem.

RANK-NULLITY THEOREM

$$\operatorname{rank}(A) + \operatorname{nullity}(A) = n$$

n is the number of columns of A. Nullity is the dimension of the null space.

Projection, orthogonal, idempotent and partitioned matrices, and quadratic forms

SYMMETRIC PLUS SKEW DECOMPOSITION

$$A = \underbrace{\tfrac{1}{2}(A + A^T)}_{\text{symmetric}} + \underbrace{\tfrac{1}{2}(A - A^T)}_{\text{skew-symmetric}}$$

Holds for every square real matrix, and the split is unique.

ORTHOGONAL MATRIX

$$Q^TQ = QQ^T = I, \qquad Q^{-1} = Q^T$$

Square real Q. Columns (and rows) form an orthonormal set.

LENGTH AND ANGLE PRESERVATION

$$\|Q\mathbf{x}\| = \|\mathbf{x}\|, \qquad (Q\mathbf{x})^T(Q\mathbf{y}) = \mathbf{x}^T\mathbf{y}$$

Holds for every $\mathbf{x}, \mathbf{y}$ when Q is orthogonal.

DETERMINANT AND EIGENVALUES OF AN ORTHOGONAL MATRIX

$$\det Q = \pm 1, \qquad |\lambda| = 1 \text{ for every eigenvalue } \lambda$$

A real eigenvalue can only be $+1$ or -1. Complex eigenvalues come in conjugate pairs of modulus 1.

EVERY 2×2 ORTHOGONAL MATRIX

$$R(\theta) = \begin{bmatrix} \cos\theta & -\sin\theta \\ \sin\theta & \cos\theta \end{bmatrix}, \qquad F(\theta) = \begin{bmatrix} \cos\theta & \sin\theta \\ \sin\theta & -\cos\theta \end{bmatrix}$$

Every real 2×2 orthogonal matrix is one of these two forms. $R(\theta)$ is a rotation by θ: $\det = +1$, eigenvalues $\cos\theta \pm i\sin\theta$. $F(\theta)$ is a reflection: symmetric, $\det = -1$, eigenvalues 1 and -1.

IDEMPOTENT: EIGENVALUES AND RANK

$$A^2 = A \;\Rightarrow\; \lambda \in \{0, 1\}, \qquad \operatorname{rank}(A) = \operatorname{trace}(A)$$

Square matrix with $A^2 = A$. The converse is false: $\operatorname{diag}(3, -1)$ has rank $2 =$ trace 2 and is not idempotent. For a general matrix, rank and trace are unrelated.

COMPLEMENT OF AN IDEMPOTENT

$$(I - A)^2 = I - A, \qquad A(I - A) = (I - A)A = 0$$

If A ($n \times n$) is idempotent, then so is $I - A$, and $\operatorname{rank}(A) + \operatorname{rank}(I - A) = n$.

PROJECTION MATRIX ONTO THE COLUMN SPACE OF A

$$P = A\,(A^TA)^{-1}A^T \qquad (P \text{ is } n \times n)$$

Needs the k columns of A linearly independent, so that the $k \times k$ matrix A^TA is invertible. $P\mathbf{b}$ is the point of the column space nearest to $\mathbf{b}$.

SPECIAL CASES OF THE PROJECTION MATRIX

$$P = \frac{\mathbf{a}\mathbf{a}^T}{\mathbf{a}^T\mathbf{a}}, \qquad P = QQ^T \;\; (Q \text{ has orthonormal columns})$$

First: projection onto the line through $\mathbf{a} \neq \mathbf{0}$. Second: Q is $n \times k$ with $Q^TQ = I_k$, and need not be square.

PROPERTIES OF AN ORTHOGONAL PROJECTION

$$P^2 = P, \quad P^T = P, \quad \operatorname{trace}(P) = \operatorname{rank}(P) = k, \quad \lambda \in \{0, 1\}$$

For $P = A(A^TA)^{-1}A^T$ with A of size $n \times k$ and rank k. P has eigenvalue 1 on the column space (multiplicity k) and 0 on its orthogonal complement (multiplicity $n - k$). P depends only on the column space, not on which basis forms the columns of A.

REFLECTION MATRIX

$$H = I - 2\,\frac{\mathbf{a}\mathbf{a}^T}{\mathbf{a}^T\mathbf{a}}, \qquad H^T = H, \quad H^TH = H^2 = I, \quad \det H = -1$$

$\mathbf{a} \neq \mathbf{0}$ in $\mathbb{R}^n$, and H is $n \times n$. $H = I - 2P$, with P the projection onto the line through $\mathbf{a}$. H reflects across the hyperplane perpendicular to $\mathbf{a}$. Its eigenvalues are -1 once (along $\mathbf{a}$) and 1 with multiplicity $n - 1$. More generally, $I - 2P$ is symmetric and orthogonal for every orthogonal projection P.

BLOCK MULTIPLICATION

$$\begin{bmatrix} A & B \\ C & D \end{bmatrix}\begin{bmatrix} E & F \\ G & H \end{bmatrix} = \begin{bmatrix} AE + BG & AF + BH \\ CE + DG & CF + DH \end{bmatrix}$$

The column partition of the left factor must match the row partition of the right factor, so that every product such as AE and BG is defined.

TRANSPOSE OF A PARTITIONED MATRIX

$$\begin{bmatrix} A & B \\ C & D \end{bmatrix}^T = \begin{bmatrix} A^T & C^T \\ B^T & D^T \end{bmatrix}$$

Transpose each block and also swap the two off-diagonal blocks.

DETERMINANT OF A BLOCK TRIANGULAR MATRIX

$$\det \begin{bmatrix} A & B \\ 0 & D \end{bmatrix} = \det A \cdot \det D$$

A and D square. The same holds for block lower triangular and block diagonal matrices. The eigenvalues of the whole matrix are those of A together with those of D, with multiplicities.

INVERSES OF BLOCK DIAGONAL AND BLOCK TRIANGULAR MATRICES

$$\begin{bmatrix} A & 0 \\ 0 & D \end{bmatrix}^{-1} = \begin{bmatrix} A^{-1} & 0 \\ 0 & D^{-1} \end{bmatrix}, \qquad \begin{bmatrix} A & B \\ 0 & D \end{bmatrix}^{-1} = \begin{bmatrix} A^{-1} & -A^{-1}BD^{-1} \\ 0 & D^{-1} \end{bmatrix}$$

A and D square. The matrix is invertible exactly when A and D are both invertible.

SCHUR COMPLEMENT DETERMINANT

$$\det \begin{bmatrix} A & B \\ C & D \end{bmatrix} = \det A \cdot \det\left(D - CA^{-1}B\right)$$

Requires A square and invertible, and D square. $D - CA^{-1}B$ is the Schur complement of A. If instead D is invertible, $\det \begin{bmatrix} A & B \\ C & D \end{bmatrix} = \det D \cdot \det(A - BD^{-1}C)$.

QUADRATIC FORM AND ITS SYMMETRIC MATRIX

$$q(\mathbf{x}) = \mathbf{x}^T A \mathbf{x} = \sum_i a_{ii} x_i^2 + 2 \sum_{i<j} a_{ij} x_i x_j, \qquad A = \tfrac{1}{2}(M + M^T)$$

A is real symmetric. Example: $x_1^2 + 6x_1x_2 + 4x_2^2$ has $A = \begin{bmatrix} 1 & 3 \\ 3 & 4 \end{bmatrix}$.

PRINCIPAL AXES (ORTHOGONAL DIAGONALISATION)

$$A = Q \Lambda Q^T, \qquad q(\mathbf{x}) = \sum_{i=1}^{n} \lambda_i \, y_i^2 \ \text{ with } \mathbf{y} = Q^T \mathbf{x}$$

A real symmetric A always has real eigenvalues λ_i and an orthogonal $n \times n$ matrix Q whose columns are eigenvectors. $\Lambda = \operatorname{diag}(\lambda_1, \ldots, \lambda_n)$. The signs of the λ_i decide the definiteness.

LEADING PRINCIPAL MINOR TEST

$$\Delta_k = \det\left(A_{1:k,\,1:k}\right) > 0 \ \text{ for } k = 1, \ldots, n \iff A \text{ positive definite}$$

A real symmetric. Δ_k is the determinant of the top-left $k \times k$ block. For $n = 2$: $\begin{bmatrix} a & b \\ b & c \end{bmatrix}$ is positive definite exactly when $a > 0$ and $ac - b^2 > 0$.

EXTREME VALUES OF A QUADRATIC FORM ON THE UNIT SPHERE

$$\lambda_{\min} \mathbf{x}^T \mathbf{x} \le \mathbf{x}^T A \mathbf{x} \le \lambda_{\max} \mathbf{x}^T \mathbf{x}, \qquad \max_{\|\mathbf{x}\|=1} \mathbf{x}^T A \mathbf{x} = \lambda_{\max}, \quad \min_{\|\mathbf{x}\|=1} \mathbf{x}^T A \mathbf{x} = \lambda_{\min}$$

A real symmetric. Equality holds when $\mathbf{x}$ is an eigenvector for $\lambda_{\max}$ (or for $\lambda_{\min}$). It follows from $q = \sum \lambda_i y_i^2$ with $\|\mathbf{y}\| = \|\mathbf{x}\|$.

Systems of linear equations, Gaussian elimination, rank and nullity

RANK AND FREE VARIABLES

$$\operatorname{rank}(A) = \text{number of pivots}, \qquad \text{number of free variables} = n - \operatorname{rank}(A)$$

Applies to any $m \times n$ matrix A after row reduction; the rank is the same in every echelon form.

ROUCHÉ-CAPELLI CONSISTENCY TEST

$$A\mathbf{x} = \mathbf{b} \text{ is consistent } \iff \operatorname{rank}(A) = \operatorname{rank}([A \mid \mathbf{b}])$$

Equivalent statement: $\mathbf{b}$ lies in the column space of A. Always $\operatorname{rank}(A) \le \operatorname{rank}([A \mid \mathbf{b}]) \le \operatorname{rank}(A) + 1$.

STRUCTURE OF THE SOLUTION SET

$$\{\mathbf{x} : A\mathbf{x} = \mathbf{b}\} = \mathbf{x}_p + N(A), \qquad N(A) = \{\mathbf{x} : A\mathbf{x} = \mathbf{0}\}$$

When the system is consistent, $\mathbf{x}_p$ is any one particular solution; the general solution adds an arbitrary null-space vector.

RANK-NULLITY THEOREM

$$\operatorname{rank}(A) + \operatorname{nullity}(A) = n$$

n is the number of columns of A (not rows). Nullity equals the number of free variables in an echelon form of A.

RANK BOUNDS

$$\operatorname{rank}(A) \le \min(m, n), \qquad \operatorname{rank}(A) = \operatorname{rank}(A^T), \qquad \operatorname{rank}(AB) \le \min\{\operatorname{rank}(A), \operatorname{rank}(B)\}$$

For A of size $m \times n$ and any product AB that is defined. For real matrices also $\operatorname{rank}(A^TA) = \operatorname{rank}(AA^T) = \operatorname{rank}(A)$.

MORE RANK FACTS

$$\operatorname{rank}(A) + \operatorname{rank}(B) - n \le \operatorname{rank}(AB), \qquad \operatorname{rank}(A+B) \le \operatorname{rank}(A) + \operatorname{rank}(B), \qquad \operatorname{rank}(PAQ) = \operatorname{rank}(A)$$

First (Sylvester's inequality): A is $m \times n$ and B is $n \times p$. Second: A and B have the same size. Third: P ($m \times m$) and Q ($n \times n$) are invertible. Also, for non-zero $\mathbf{u} \in \mathbb{R}^m$ and $\mathbf{v} \in \mathbb{R}^n$, the $m \times n$ matrix $\mathbf{u}\mathbf{v}^T$ has rank 1.

INVERSE BY ROW REDUCTION

$$[A \mid I] \xrightarrow{\text{row operations}} [I \mid A^{-1}]$$

For a square $n \times n$ matrix, with I the $n \times n$ identity. If the left block cannot be reduced to I (a zero row appears in it), A is singular and has no inverse.

CRAMER'S RULE

$$x_i = \frac{\det(A_i)}{\det(A)}, \qquad i = 1, \dots, n$$

For a square $n \times n$ system with $\det A \neq 0$. A_i is A with its i-th column replaced by $\mathbf{b}$.

Eigenvalues, eigenvectors and determinants

DETERMINANT OF A 2 BY 2 MATRIX

$$\det \begin{bmatrix} a & b \\ c & d \end{bmatrix} = ad - bc$$

Always valid for a 2 by 2 matrix.

COFACTOR EXPANSION ALONG ROW I

$$\det(A) = \sum_{j=1}^{n} a_{ij}\,(-1)^{i+j} M_{ij}$$

A is n by n and i is any fixed row. M_ij is the determinant of the (n-1) by (n-1) matrix left after deleting row i and column j. The same works along any column. Expand along the row or column with the most zeros.

PRODUCT, TRANSPOSE AND INVERSE

$$\det(AB) = \det(A)\det(B), \qquad \det(A^T) = \det(A), \qquad \det(A^{-1}) = \frac{1}{\det(A)}$$

A and B are n by n. The inverse rule needs det(A) not equal to 0. Applying the product rule repeatedly gives det(A^k) = (det A)^k for every integer k >= 1.

SCALAR MULTIPLE

$$\det(cA) = c^{n}\det(A)$$

A is n by n. The factor c comes out of each of the n rows, so the power is n, the size of the matrix.

ADJOINT (ADJUGATE)

$$A\,\operatorname{adj}(A) = \operatorname{adj}(A)\,A = \det(A)\,I, \qquad \det\big(\operatorname{adj}(A)\big) = \big(\det A\big)^{n-1}$$

A is n by n with n >= 2, and adj(A) is the transpose of the cofactor matrix. If det(A) is not 0, then A^{-1} = adj(A)/det(A). Also adj(adj A) = (det A)^{n-2} A for n >= 3, and adj(adj A) = A for n = 2.

VANDERMONDE DETERMINANT

$$\det\begin{bmatrix} 1 & x_1 & x_1^2 \\ 1 & x_2 & x_2^2 \\ 1 & x_3 & x_3^2 \end{bmatrix} = (x_2 - x_1)(x_3 - x_1)(x_3 - x_2)$$

In general, the n by n matrix whose i-th row is 1, x_i, x_i^2, ..., x_i^{n-1} has determinant equal to the product of (x_j - x_i) over all i < j. It is non-zero exactly when the x_i are distinct.

CHARACTERISTIC EQUATION

$$\det(A - \lambda I) = 0$$

A is n by n. The n roots of this equation, counted with multiplicity over the complex numbers, are the eigenvalues. For each root, solve (A - lambda I)x = 0 to find the eigenvectors.

CHARACTERISTIC EQUATION OF A 2 BY 2 MATRIX

$$\lambda^2 - \operatorname{tr}(A)\,\lambda + \det(A) = 0$$

Valid for any 2 by 2 matrix; it is det(A - lambda I) = 0 written out.

CHARACTERISTIC EQUATION OF A 3 BY 3 MATRIX

$$\lambda^3 - \operatorname{tr}(A)\,\lambda^2 + c_2\,\lambda - \det(A) = 0$$

c_2 is the sum of the three principal 2 by 2 minors: for each i, delete row i and column i, take the determinant, and add the three results. This is det(A - lambda I) = 0 multiplied through by -1.

TRACE AND DETERMINANT FROM THE EIGENVALUES

$$\sum_{i=1}^{n} \lambda_i = \operatorname{tr}(A), \qquad \prod_{i=1}^{n} \lambda_i = \det(A)$$

A is n by n, and the eigenvalues are counted with algebraic multiplicity. Hence A is singular if and only if 0 is an eigenvalue. Applying the trace rule to A^k gives tr(A^k) = lambda_1^k + ... + lambda_n^k for every integer k >= 1.

DIAGONALISATION

$$A = PDP^{-1}, \qquad D = \operatorname{diag}(\lambda_1, \ldots, \lambda_n), \quad A^k = PD^kP^{-1}$$

A, P and D are n by n and P is invertible. The columns of P are n linearly independent eigenvectors, in the same order as the eigenvalues on the diagonal of D. This factorisation exists if and only if A has n linearly independent eigenvectors.

SPECTRAL THEOREM FOR REAL SYMMETRIC MATRICES

$$A = Q\Lambda Q^T, \qquad Q^TQ = I$$

A is an n by n real symmetric matrix, Q is an n by n orthogonal matrix whose columns are orthonormal eigenvectors, and Lambda is the diagonal matrix of the matching eigenvalues. A real symmetric matrix has real eigenvalues, its eigenvectors for distinct eigenvalues are orthogonal, and it is always diagonalisable by an orthogonal matrix.

RAYLEIGH QUOTIENT BOUNDS

$$\lambda_{\min} \le \frac{\mathbf{x}^TA\mathbf{x}}{\mathbf{x}^T\mathbf{x}} \le \lambda_{\max} \qquad (\mathbf{x} \neq \mathbf{0})$$

A is an n by n real symmetric matrix with smallest eigenvalue lambda_min and largest eigenvalue lambda_max. Each bound is attained at an eigenvector for that eigenvalue, so the maximum of x^T A x over unit vectors (||x|| = 1) is lambda_max and the minimum is lambda_min.

CAYLEY-HAMILTON THEOREM

$$p(\lambda) = \det(A - \lambda I) \implies p(A) = 0$$

A is n by n, and the constant term c of p is replaced by cI. Every square matrix satisfies its own characteristic equation. For a 2 by 2 matrix: A^2 - tr(A) A + det(A) I = 0.

Projections, LU decomposition and singular value decomposition

PROJECTION ONTO A LINE

$$\mathbf{p} = \frac{\mathbf{a}^T\mathbf{b}}{\mathbf{a}^T\mathbf{a}}\,\mathbf{a}, \qquad P = \frac{\mathbf{a}\mathbf{a}^T}{\mathbf{a}^T\mathbf{a}}$$

Line through the origin along $\mathbf{a} \neq \mathbf{0}$; $\mathbf{b}$ and $\mathbf{a}$ are in $\mathbb{R}^n$, P is $n \times n$ of rank 1, and $\mathbf{p} = P\mathbf{b}$.

PROJECTION ONTO A COLUMN SPACE

$$P = A\,(A^TA)^{-1}A^T, \qquad \mathbf{p} = P\mathbf{b}$$

A is $m \times n$ with linearly independent columns (so A^TA is invertible); P is $m \times m$ and projects onto the column space of A.

PROJECTION WITH AN ORTHONORMAL BASIS

$$P = QQ^T = \sum_{i=1}^{k} \mathbf{q}_i\mathbf{q}_i^T$$

The columns $\mathbf{q}_1, \ldots, \mathbf{q}_k$ of the $m \times k$ matrix Q are orthonormal ($Q^TQ = I_k$); P is $m \times m$ and no inverse is needed.

NORMAL EQUATIONS

$$A^TA\,\hat{\mathbf{x}} = A^T\mathbf{b} \quad \implies \quad \hat{\mathbf{x}} = (A^TA)^{-1}A^T\mathbf{b}$$

The normal equations always have a solution; it is unique (and given by the inverse form) only when the columns of A are linearly independent.

RESIDUAL IS ORTHOGONAL TO THE COLUMNS

$$A^T(\mathbf{b} - A\hat{\mathbf{x}}) = \mathbf{0}, \qquad \|\mathbf{b}\|^2 = \|A\hat{\mathbf{x}}\|^2 + \|\mathbf{b} - A\hat{\mathbf{x}}\|^2$$

Pythagoras applies because the projection and the residual are perpendicular.

GRAM-SCHMIDT STEP

$$\mathbf{v}_1 = \mathbf{a}_1, \qquad \mathbf{v}_j = \mathbf{a}_j - \sum_{i<j} \frac{\mathbf{v}_i^T \mathbf{a}_j}{\mathbf{v}_i^T \mathbf{v}_i} \mathbf{v}_i, \qquad \mathbf{q}_j = \frac{\mathbf{v}_j}{\|\mathbf{v}_j\|}$$

For an $m \times k$ matrix A with linearly independent columns this gives $A = QR$: Q is $m \times k$ with orthonormal columns and $R = Q^T A$ is $k \times k$ upper triangular with entries $r_{ij} = \mathbf{q}_i^T \mathbf{a}_j$ ($i \le j$) and positive diagonal. The span of the first j vectors is preserved at every step.

LU FACTORISATION

$$A = LU, \qquad L = \begin{bmatrix} 1 & 0 & 0 \\ \ell_{21} & 1 & 0 \\ \ell_{31} & \ell_{32} & 1 \end{bmatrix}, \quad U = \begin{bmatrix} u_{11} & u_{12} & u_{13} \\ 0 & u_{22} & u_{23} \\ 0 & 0 & u_{33} \end{bmatrix}$$

Shown for 3×3. Elimination without row exchanges succeeds exactly when the first $n-1$ pivots are non-zero, equivalently when the leading principal minors $\Delta_1, \ldots, \Delta_{n-1}$ are non-zero; the factorisation with unit lower L is then unique. (u_{nn} may be 0, which happens exactly when A is singular.)

PIVOTS FROM LEADING PRINCIPAL MINORS

$$u_{kk} = \frac{\Delta_k}{\Delta_{k-1}}, \qquad \Delta_0 = 1$$

Δ_k is the determinant of the top-left $k \times k$ block of A; valid when no row exchanges are needed ($\Delta_1, \ldots, \Delta_{n-1} \neq 0$).

PIVOTED FORM

$$PA = LU$$

When a zero pivot appears, rows are exchanged; P is an $n \times n$ permutation matrix. A factorisation $PA = LU$ exists for every square A, and $\det A = (-1)^s \prod_i u_{ii}$, where s is the number of row exchanges.

DETERMINANT FROM LU

$$\det A = \det L \cdot \det U = \prod_{i=1}^{n} u_{ii}$$

Valid when no row exchanges were used; $\det L = 1$ because L is unit triangular.

SYMMETRIC MATRICES: LDL^T AND CHOLESKY

$$A = LDL^T, \quad D = \operatorname{diag}(u_{11}, \ldots, u_{nn}); \qquad A = GG^T, \quad G = LD^{1/2}$$

A is symmetric $n \times n$ with $\Delta_1, \ldots, \Delta_n \neq 0$; then $U = DL^T$. A is positive definite exactly when every pivot $u_{kk} > 0$ (equivalently every $\Delta_k > 0$), and then the Cholesky factor G is lower triangular with positive diagonal.

SINGULAR VALUE DECOMPOSITION

$$A = U \Sigma V^T, \qquad \Sigma = \operatorname{diag}(\sigma_1, \ldots, \sigma_r, 0, \ldots, 0), \quad \sigma_1 \ge \sigma_2 \ge \cdots \ge \sigma_r > 0$$

U is $m \times m$ orthogonal (left singular vectors $\mathbf{u}_i$), V is $n \times n$ orthogonal (right singular vectors $\mathbf{v}_i$), Σ is $m \times n$ with the $\min(m, n)$ singular values on its diagonal and zeros elsewhere; $r = \operatorname{rank} A$. Equivalently $A = \sum_{i=1}^{r} \sigma_i \mathbf{u}_i \mathbf{v}_i^T$.

SINGULAR VALUES FROM EIGENVALUES

$$\sigma_i = \sqrt{\lambda_i(A^T A)} = \sqrt{\lambda_i(A A^T)} \quad (i = 1, \ldots, r), \qquad A^T A \, \mathbf{v}_i = \sigma_i^2 \mathbf{v}_i$$

$A^T A$ ($n \times n$) and $A A^T$ ($m \times m$) are symmetric positive semidefinite and have the same non-zero eigenvalues; all their other eigenvalues are 0. The columns of V are orthonormal eigenvectors of $A^T A$ and the columns of U are orthonormal eigenvectors of $A A^T$.

LEFT VECTORS FROM RIGHT VECTORS

$$\mathbf{u}_i = \frac{1}{\sigma_i} A \mathbf{v}_i \qquad (\sigma_i > 0)$$

Fixes $\mathbf{u}_i$ so that its sign agrees with $\mathbf{v}_i$; choosing $\mathbf{u}_i$ independently from $A A^T$ can give a wrong sign.

RANK-ONE MATRIX

$$A = \mathbf{u}\mathbf{v}^T \implies \sigma_1 = \|\mathbf{u}\|\,\|\mathbf{v}\|, \quad \sigma_2 = \cdots = 0$$

$\mathbf{u} \in \mathbb{R}^m$ and $\mathbf{v} \in \mathbb{R}^n$ are non-zero, so A is $m \times n$ of rank 1; this follows from $A^TA = \|\mathbf{u}\|^2\mathbf{v}\mathbf{v}^T$.

RANK AND NORMS FROM SINGULAR VALUES

$$\operatorname{rank} A = \#\{\sigma_i > 0\}, \quad \|A\|_2 = \sigma_1, \quad \|A\|_F^2 = \sum_i \sigma_i^2 = \operatorname{trace}(A^TA)$$

$\|A\|_2 = \max_{\|\mathbf{x}\|=1}\|A\mathbf{x}\|$ is the spectral norm; $\|A\|_F$ is the Frobenius norm (square root of the sum of squared entries).

SQUARE MATRICES

$$|\det A| = \prod_{i=1}^{n} \sigma_i, \qquad \kappa_2(A) = \frac{\sigma_1}{\sigma_n} \ \ (A \text{ invertible})$$

A is $n \times n$; A is invertible exactly when $\sigma_n > 0$, and κ_2 is the 2-norm condition number.

RANK-K TRUNCATION

$$A_k = \sum_{i=1}^{k} \sigma_i\, \mathbf{u}_i\mathbf{v}_i^T$$

Keeps the k largest singular values, $k \le r$; A_k has rank k.

BEST LOW-RANK APPROXIMATION (ECKART-YOUNG)

$$\min_{\operatorname{rank} B \le k} \|A - B\|_2 = \sigma_{k+1}, \qquad \min_{\operatorname{rank} B \le k} \|A - B\|_F = \sqrt{\sum_{i>k} \sigma_i^2}$$

B ranges over $m \times n$ matrices; the minimum is attained by $B = A_k$ in both norms; take $\sigma_{k+1} = 0$ if $k \ge r$.

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