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General Aptitude Formula Sheet

INSIDE

- 223 formulas in 8 topics
- When to use each one
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Grammar and vocabulary

PRESENT PERFECT

$$\text{Subject} + \text{has / have} + \text{V3}$$

A past action with a present result, or a period that runs up to now. Never with a finished past time (yesterday, ago, last year, in 2019).

PRESENT PERFECT CONTINUOUS

$$\text{has / have been} + \text{V-ing} + \text{since (point) or for (period)}$$

An action that began in the past and is still going on: 'She has been teaching here since June' / 'for three months'. It is also used for an activity that has only just stopped, leaving a visible result: 'The roads are wet; it has been raining.' Use since or for here, not from.

PAST PERFECT

$$\text{had} + \text{V3 (earlier action)} \quad \text{past simple (later action)}$$

Marks an action completed before another past action or a stated past time: 'The train had left when we reached the platform.' Not for a lone past event: 'I went there yesterday', not 'I had gone there yesterday'.

FUTURE PERFECT

$$\text{will have} + \text{V3} \quad (\text{by} + \text{future time})$$

An action that will be complete before a stated future point: 'By March, the team will have tested every unit.'

FUTURE TIME IN TIME AND CONDITION CLAUSES

$$\text{when / if / until / unless / as soon as / by the time} + \text{present simple (or present perfect)}$$

These clauses (also before, after, once) take the present simple, or the present perfect for an action that will be complete, even for future time; 'will' goes only in the main clause: 'I will call you when I reach the station', not 'when I will reach the station'; 'You may leave once you have signed the register.' The rule covers clauses of time and condition only. A when- or if-clause that is the object of a verb such as know, ask, tell or wonder is a noun clause and takes 'will' as usual: 'Nobody knows when the results will be declared.' 'If you will' is also correct where 'will' means willingness, as in a polite request: 'If you will wait here, I will fetch the form.'

ZERO CONDITIONAL

$$\text{If / When} + \text{present simple}, \quad \text{present simple}$$

A general truth or a rule that always holds: 'If you mix blue and yellow paint, you get green.'

FIRST CONDITIONAL

$$\text{If} + \text{present simple}, \quad \text{will} + \text{V1}$$

A real, possible future: 'If the supply fails, the backup will start.'

SECOND CONDITIONAL

$$\text{If} + \text{past simple}, \quad \text{would / could / might} + \text{V1}$$

An unreal present or an unlikely future. Formal English uses 'were' for every person: 'If I were the organiser, I would start earlier.'

THIRD CONDITIONAL

$$\text{If} + \text{had} + \text{V3}, \quad \text{would / could / might have} + \text{V3}$$

An unreal past. 'Would have' never goes in the if-clause. Inverted form without 'if': 'Had I known, I would have helped.' A mixed conditional joins an unreal past condition to a present result: 'If I had taken that job, I would be living in Pune now.'

SEQUENCE OF TENSES

$$\text{past reporting verb} \;\rightarrow\; \text{past form in the that-clause}$$

'She said that she was tired.' In reported speech after a past reporting verb, each tense moves one step back: is → was, is doing → was doing, has done → had done, did → had done (often left as did when the order of events is clear), will → would, can → could, may → might. A universal truth keeps the present: 'The teacher said that light travels faster than sound.' After a present or future reporting verb nothing shifts: 'She says that she is tired.'

PASSIVE VOICE

$$\text{be (in the tense of the active verb)} + \text{V3 (+ by + agent)}$$

Only transitive verbs have a passive. Happen, occur, arrive, die, seem and other intransitive verbs cannot be made passive: 'The accident happened', never 'was happened'. Not every transitive verb has one either: have (in the sense of own), resemble, lack and suit are not normally passive ('Ravi resembles his father', never 'His father is resembled by Ravi').

PROXIMITY RULE

$$\text{either } X \text{ or } Y,\;\; \text{neither } X \text{ nor } Y,\;\; \text{not only } X \text{ but also } Y \;\rightarrow\; \text{verb agrees with } Y$$

Also for plain 'X or Y'. The verb agrees with the nearer subject: 'Neither the lecturer nor the students were informed.' 'Either the students or the lecturer is responsible.'

ADDED-PHRASE RULE

$$X \text{ along with / as well as / together with / besides } Y \;\rightarrow\; \text{verb agrees with } X$$

Also 'in addition to', 'accompanied by', 'like', 'including'. 'The captain, along with his players, was felicitated.' Of the joining words, only 'and' makes a plural subject, and even 'and' does not when the two nouns name one person or one idea (see the list below).

ONE OF THE ...

$$\text{One of the + plural noun + singular verb}$$

'One of the valves is faulty.' But in 'one of the + plural noun + who/that + verb', the relative clause refers to the plural noun, so its verb is plural: 'She is one of the engineers who have designed the dam.' With 'the only one of ... who', it is singular again.

THE NUMBER / A NUMBER

$$\text{The number of + plural noun} \rightarrow \text{singular};\quad \text{A number of + plural noun} \rightarrow \text{plural}$$

'The number of applicants has doubled.' 'A number of applicants have complained.'

EACH, EVERY, EITHER, NEITHER

$$\text{each / every / either / neither / many a + singular noun} \rightarrow \text{singular verb}$$

'Each of / Either of / Neither of + plural noun' is also singular in formal English: 'Neither of the answers is correct.' Exception: 'each' AFTER a plural subject keeps the verb plural: 'The boys each have a locker.'

ORDER OF ADJECTIVES BEFORE A NOUN

$$\text{Opinion} \rightarrow \text{Size} \rightarrow \text{Age} \rightarrow \text{Shape} \rightarrow \text{Colour} \rightarrow \text{Origin} \rightarrow \text{Material} \rightarrow \text{Purpose} \rightarrow \text{Noun}$$

Articles, possessives and numbers come first. 'a beautiful old Kashmiri woollen shawl' (opinion, age, origin, material); 'a large round wooden table' (size, shape, material).

LATIN COMPARATIVES

$$\text{senior / junior / superior / inferior / prior / preferable + to}$$

These take 'to', never 'than', and never 'more' in front ('more superior' and 'more preferable' are wrong): 'She is senior to me by two years.' Likewise the verb 'prefer X to Y': 'He prefers trains to buses.' With infinitives, use 'rather than': 'He prefers to walk rather than wait.'

PARALLEL CORRELATIVE PAIRS

$$\text{both } A \text{ and } B, \quad \text{either } A \text{ or } B, \quad \text{not only } A \text{ but also } B \;:\; A, B \text{ in the same form}$$

Each half stands directly before matching words: noun with noun, verb with verb. 'She is not only a singer but also a dancer.' 'He not only designs circuits but also tests them.' Also whether ... or, neither ... nor.

NO SOONER / HARDLY / SCARCELY

$$\text{No sooner had} + \text{subject} + \text{V3} + \text{than}; \quad \text{Hardly / Scarcely had} + \text{subject} + \text{V3} + \text{when}$$

Inverted order after the negative opener. 'No sooner had the lecture begun than the lights went out.' 'Hardly had we sat down when the bell rang.' 'Before' may replace 'when' after hardly and scarcely, and the past simple form 'No sooner did + subject + V1 ... than' is also correct: 'No sooner did the lecture begin than the lights went out.' Never 'No sooner ... when' or 'Hardly ... than'.

Reading comprehension and narrative sequencing

THE INFERENCE TEST

$$\text{Inferable} \;\leftrightarrow\; \text{cannot be false if the passage is true}$$

Here $\leftrightarrow$ is read 'means exactly the same as'. For 'can be inferred', 'is implied' and 'must be true', an option passes only if it holds in every situation in which all the passage's sentences hold. 'Probably true' and 'true in real life' do not pass. Ordinary word meanings may be used in applying the test; facts about the topic that the passage does not give may not.

CONTRAPOSITIVE

$$\text{if } A \text{ then } B \;\leftrightarrow\; \text{if not } B \text{ then not } A$$

Of the three standard rearrangements of an if-then statement (converse, inverse, contrapositive), only the contrapositive keeps the meaning. The converse (if B then A) and the inverse (if not A then not B) do NOT follow; they are equivalent to each other, not to the original.

'IF' (SUFFICIENT CONDITION)

$$A \text{ if } B \;\leftrightarrow\; \text{if } B \text{ then } A$$

The part that follows 'if' is sufficient: it guarantees the other part. 'Only if', below, points the other way. 'A if and only if B' asserts both directions: if B then A, and if A then B.

'ONLY IF' (NECESSARY CONDITION)

$$A \text{ only if } B \;\leftrightarrow\; \text{if } A \text{ then } B$$

B is necessary for A. Knowing that B holds tells you nothing about A; knowing that B fails tells you that A fails.

'ONLY'

$$\text{only } X \text{ are } Y \;\leftrightarrow\; \text{every } Y \text{ is an } X$$

The group named after 'only' becomes the one that every member of the other group belongs to. It does not say that every X is a Y.

'UNLESS'

$$A \text{ unless } B \;\leftrightarrow\; \text{if not } B \text{ then } A$$

Replace 'unless' with 'if not'. Its contrapositive, 'if not A then B', is the same statement. On its own, the sentence says nothing about the case where B occurs.

'SOME' (REASONING CONVENTION)

$$\text{some } X \text{ are } Y \;\leftrightarrow\; \text{at least one } X \text{ is a } Y$$

'Some' means at least one and possibly all. It does not by itself mean 'not all'. 'At least' and 'at most' are read inclusively.

NEGATION OF 'ALL'

$$\text{not (all } X \text{ are } Y) \;\leftrightarrow\; \text{at least one } X \text{ is not a } Y$$

The opposite of 'all' is 'not all', not 'none'. You need this when you apply the negation test to an option that contains 'all' or 'every'.

NEGATION OF 'SOME'

$$\text{not (some } X \text{ are } Y) \;\leftrightarrow\; \text{no } X \text{ is a } Y$$

The opposite of 'at least one' is 'none'. Read the other way, the opposite of 'no X is a Y' is 'at least one X is a Y'.

NEGATION OF 'AND'

$$\text{not } (A \text{ and } B) \;\leftrightarrow\; \text{not } A \text{ or not } B$$

Denying 'both' says only that at least one of the two fails. It does not say that both fail.

NEGATION OF 'OR'

$$\text{not } (A \text{ or } B) \;\leftrightarrow\; \text{not } A \text{ and not } B$$

Denying 'either' (inclusive or) says that both fail. This is what 'neither A nor B' means.

NEGATION TEST FOR AN ASSUMPTION

$$\text{not(option)} \;\rightarrow\; \text{conclusion no longer follows}$$

Here $\rightarrow$ is read 'leads to'. State the mildest opposite of the option (the opposite of 'all' is 'not all'). If the conclusion then no longer follows from the evidence, the option is an assumption the argument needs. If the argument survives, it is not an assumption.

Data interpretation

PERCENTAGE SHARE

$$\text{Share of } A = \frac{A}{\text{Total}} \times 100\%$$

For parts of one whole (a pie, a stacked bar, a column of a table with its total). The shares of all the parts add to 100%, apart from rounding in the display.

PERCENTAGE CHANGE

$$\text{Percentage change} = \frac{\text{New value} - \text{Old value}}{\text{Old value}} \times 100$$

The base is the old (earlier) value unless the question names another. A positive result is an increase and a negative one a decrease. For a quantity that cannot go below zero, a decrease can never exceed 100%; an increase can (a rise of 100% is a doubling).

PERCENTAGE-POINT CHANGE

$$\text{Change} = p_2 - p_1 \text{ percentage points}$$

For a change in a quantity that is itself a percentage (a share, a pass rate, an interest rate). From 20% to 25% is a rise of 5 percentage points, but a 25% increase in the rate, because $\frac{25-20}{20} = 0.25$.

MORE THAN AND LESS THAN

$$A \text{ is } \frac{A-B}{B} \times 100\% \text{ more than } B, \qquad B \text{ is } \frac{A-B}{A} \times 100\% \text{ less than } A$$

For positive A and B with $A > B$. The base is the quantity after 'than'. The two percentages differ: with $A = 125$ and $B = 100$, A is 25% more than B but B is 20% less than A.

CONVERTING 'MORE THAN' INTO 'LESS THAN', AND RECOVERING A FALL

$$r\%\text{ more }\Rightarrow\ \frac{100r}{100+r}\%\text{ less},\qquad \text{a fall of } r\%\text{ needs a rise of }\frac{100r}{100-r}\%\text{ to recover}$$

The first holds for any $r \geq 0$; the second for $0 \leq r < 100$. A 25% rise is undone by a 20% fall, and a 20% fall needs a 25% rise to recover.

SUCCESSIVE PERCENTAGE CHANGES

$$\text{Final} = \text{Initial} \times \left(1+\frac{a}{100}\right)\left(1+\frac{b}{100}\right)\left(1+\frac{c}{100}\right)\cdots$$

Each change acts on the value left by the previous one. Take a decrease as negative. The order of the changes does not affect the final value.

NET EFFECT OF TWO SUCCESSIVE CHANGES

$$\text{Net percentage change} = a + b + \frac{ab}{100}$$

For changes of $a\%$ then $b\%$, decreases negative. A rise of 20% followed by a fall of 20% gives $20 - 20 - \frac{400}{100} = -4$, a net fall of 4%. The same result gives the change in a product: price up 10% and quantity sold down 10% changes revenue by $10 - 10 - 1 = -1$, a 1% fall.

ORIGINAL VALUE FROM A CHANGED VALUE (REVERSE PERCENTAGE)

$$\text{Original} = \frac{\text{New}}{1+\dfrac{r}{100}}$$

When the value after a change of $r\%$ is known (r negative for a decrease, with $r > -100$). A value of 132 after a 10% rise was $\frac{132}{1.1} = 120$, not $132 - 13.2 = 118.8$.

PROFIT PERCENTAGE FROM INCOME AND EXPENDITURE

$$\text{Profit percentage} = \frac{\text{Income} - \text{Expenditure}}{\text{Expenditure}} \times 100$$

The base is the expenditure (the cost) unless the question names another; a negative result is a loss. Rearranged, Income $=$ Expenditure $\times \left(1+\frac{p}{100}\right)$ for a profit of $p\%$. An income of Rs. 69 lakh on an expenditure of Rs. 60 lakh is a profit of $\frac{9}{60} \times 100 = 15\%$.

SHARE OF A SHARE

$$p\%\text{ of }q\%\text{ of the total} = \frac{pq}{100}\%\text{ of the total}$$

For nested data: if 40% of a firm's staff are engineers and 25% of the engineers are women, women engineers are $\frac{40\times 25}{100} = 10\%$ of the staff.

RATIO OF TWO SHARES OF THE SAME TOTAL

$$\frac{A}{B} = \frac{p_A}{p_B}$$

When A and B are $p_A\%$ and $p_B\%$ of the SAME total, their ratio needs no total. It fails in general across two different totals (two pies, two years): convert to amounts first.

DIVIDING A TOTAL IN A GIVEN RATIO

$$\text{Part} = \frac{a}{a+b+c} \times \text{Total}\qquad (\text{parts in the ratio } a:b:c)$$

Each part is its term of the ratio over the sum of the terms. Rs. 9 lakh divided in the ratio $2:3:4$ gives Rs. 2 lakh, Rs. 3 lakh and Rs. 4 lakh.

SECTOR ANGLE

$$\theta = \frac{\text{Part}}{\text{Total}} \times 360^\circ = p \times 3.6^\circ$$

p is the part's percentage share. 1% corresponds to 3.6°, 10% to 36°, 25% to 90° and 50% to 180°.

VALUE AND PERCENTAGE FROM AN ANGLE

$$\text{Part} = \frac{\theta}{360^\circ} \times \text{Total}, \qquad p = \frac{\theta}{3.6^\circ}$$

Needs the total of THAT pie. The angles of all the sectors add to 360° and their percentages to 100.

RATIO OF TWO SECTORS OF ONE PIE

$$\frac{\text{Part}_1}{\text{Part}_2} = \frac{\theta_1}{\theta_2} = \frac{p_1}{p_2}$$

Only within one pie. Across two pies, convert each sector to an amount using its own pie's total first. The difference between two sectors is $\frac{\theta_1 - \theta_2}{360^\circ} \times \text{Total}$.

SEGMENT OF A STACKED BAR

$$\text{Segment} = (\text{reading at its upper edge}) - (\text{reading at its lower edge})$$

The bottom segment's lower edge is zero. The upper edge of any higher segment is a running total, not that segment's value. The top of the whole stack is the total of all segments.

AMOUNT FROM A 100% STACKED BAR

$$\text{Amount} = \frac{\text{Segment percentage}}{100} \times \text{Bar total}$$

The bar totals must come from elsewhere (a table or a label). Without them, segments of different bars can be compared only as shares.

AVERAGE RATE OF CHANGE

$$\text{Average rate of change} = \frac{V_2 - V_1}{t_2 - t_1}$$

In units per period (for example tonnes per year). On a line graph it is the slope of the segment joining the two points, computed from the axis values. How steep a segment looks depends on the scales chosen for the axes.

YEAR-ON-YEAR GROWTH RATE

$$g_t = \frac{V_t - V_{t-1}}{V_{t-1}} \times 100$$

The percentage change over one period, on the earlier value as base. A falling growth rate does not mean a falling value: a line that keeps rising but flattens has positive, shrinking growth rates.

COMPOUND GROWTH AT A CONSTANT RATE

$$V_n = V_0 \left(1 + \frac{r}{100}\right)^n$$

Growth of $r\%$ in each of n periods, each period's growth applied to the value at the start of that period. For a steady decline, r is negative.

COMPOUND ANNUAL GROWTH RATE (CAGR)

$$\text{CAGR} = \left[\left(\frac{V_n}{V_0}\right)^{1/n} - 1\right] \times 100\%$$

For positive V_0 and V_n. n is the number of periods BETWEEN the two values: from 2020 to 2025, $n = 5$, not 6. CAGR is the constant rate that would carry V_0 to V_n; it ignores the path in between.

CAGR FROM THE YEARLY RATES

$$1+\frac{\text{CAGR}}{100}=\left[\left(1+\frac{g_1}{100}\right)\left(1+\frac{g_2}{100}\right)\cdots\left(1+\frac{g_n}{100}\right)\right]^{1/n}$$

The CAGR comes from the geometric mean of the growth factors. Provided every yearly rate is above -100% (no value falls to zero), it never exceeds the arithmetic mean of the yearly rates $g_1,\ldots,g_n$, and the two are equal only when every yearly rate is the same.

INDEX NUMBER

$$I_t=\frac{V_t}{V_{\text{base}}}\times 100,\qquad \text{percentage change}=\frac{I_2-I_1}{I_1}\times 100$$

The base period has index 100. The difference I_2-I_1 is a percentage change only when the earlier period is the base ($I_1=100$). Indices of 125 and 150 differ by 25 points but mean a rise of $\frac{25}{125}\times 100=20\%$.

PER-PERIOD VALUE FROM CUMULATIVE DATA

$$x_t=C_t-C_{t-1}$$

When a graph or table shows running totals C_t ('total sales up to month t'), the value for period t is the difference of consecutive readings. A running total of non-negative amounts never falls, so a line that falls anywhere is not a cumulative line of such amounts.

ARITHMETIC MEAN

$$\bar{x}=\frac{x_1+x_2+\cdots+x_n}{n}=\frac{1}{n}\sum_{i=1}^{n}x_i$$

Each item counts equally. Rearranged, sum of values $=n\bar{x}$, which recovers a missing value when the mean of all the items is given.

WEIGHTED MEAN

$$\bar{x}_w=\frac{\sum w_i x_i}{\sum w_i}$$

The weights w_i are the group sizes (students, units sold, workers). With positive weights, the weighted mean always lies between the smallest and largest x_i, pulled towards the values with larger weights. For a frequency table the weights are the frequencies; for grouped classes, taking each class mid-point as x_i gives an estimate of the mean, not its exact value.

COMBINED PERCENTAGE OF SEVERAL GROUPS

$$p_{\text{overall}}=\frac{\sum n_i p_i}{\sum n_i}$$

p_i is group i's percentage and n_i its size. The plain mean of the p_i equals p_{overall} whenever all the n_i are equal; otherwise it generally differs.

TWO-GROUP WEIGHTED MEAN AND ALLIGATION

$$\bar{x}_w=x_1+\frac{w_2}{w_1+w_2}\left(x_2-x_1\right),\qquad \frac{w_1}{w_2}=\frac{x_2-\bar{x}_w}{\bar{x}_w-x_1}$$

The combined mean divides the gap between the two group values in the inverse ratio of the weights. The second form finds the ratio of group sizes from a known combined mean and needs $x_1<\bar{x}_w<x_2$. Groups with means 60 and 80 in sizes $3:1$ have combined mean $60+\frac{1}{4}(80-60)=65$.

RATIO OF TOTALS VERSUS AVERAGE OF RATIOS

$$\text{Overall ratio}=\frac{\sum a_i}{\sum b_i},\qquad \text{which in general}\neq\frac{1}{n}\sum\frac{a_i}{b_i}$$

An overall rate (revenue per unit sold, output per worker, marks per student) is total divided by total. In general the mean of the individual ratios is a different number; the two agree when all the b_i are equal.

FREQUENCY FROM A HISTOGRAM WITH UNEQUAL CLASSES

$$\text{Frequency} = \text{Frequency density} \times \text{Class width}$$

A bar of height 6 per unit over the class 20 to 30 stands for $6 \times 10 = 60$ items. Comparing the heights of bars of different widths compares densities, not frequencies.

RATIO FROM A POINT OF A SCATTER PLOT

$$\frac{y}{x} = \text{slope of the line from } (0,0) \text{ to } (x,y)$$

Holds only when the plotted origin really is $(0,0)$. If either axis is offset (starts above zero), compute $\frac{y}{x}$ from the values read off instead.

AVERAGE GRADIENT FROM A CONTOUR MAP

$$\text{Average gradient} = \frac{\text{change in } z}{\text{horizontal distance}} = \frac{(\text{number of intervals crossed}) \times (\text{contour interval})}{\text{horizontal distance}}$$

Both in the same unit. The horizontal distance is the ground distance, so convert the map distance with the scale first. Often written as '1 in N', meaning a rise of 1 unit for every N units horizontally.

DISTANCE ON A MAP

$$\text{Ground distance} = \text{Map distance} \times k \qquad (\text{scale } 1 : k)$$

Convert units last: 1 km $= 1{,}000$ m $= 1{,}00{,}000$ cm.

AREA ON A MAP

$$\text{Ground area} = \text{Map area} \times k^2$$

Areas scale with the SQUARE of the linear scale. On a $1 : 50{,}000$ map, 1 cm represents 0.5 km, so 1 cm² represents $(0.5)^2 = 0.25$ km².

Numerical computation: ratios, percentages, powers, logarithms, counting and series

COMBINING TWO RATIOS THROUGH A COMMON TERM

$$a : b = p : q, \quad b : c = r : s \;\Rightarrow\; a : b : c = pr : qr : qs$$

Scale both ratios so that the common term b has the same value (qr) in each.

DIVIDING N IN THE RATIO A : B : C

$$\text{share of } A = \frac{a}{a+b+c} \times N, \quad \text{share of } B = \frac{b}{a+b+c} \times N, \quad \text{share of } C = \frac{c}{a+b+c} \times N$$

The shares add back to N. If only the difference between two shares is given, divide it by the difference of their ratio terms to get the value of one ratio unit.

PROPORTION; COMPONENDO AND DIVIDENDO

$$\frac{a}{b} = \frac{c}{d} \;\Leftrightarrow\; ad = bc, \qquad \frac{a}{b} = \frac{c}{d} \;\Rightarrow\; \frac{a+b}{a-b} = \frac{c+d}{c-d}$$

Needs $b, d \neq 0$, and $a \neq b$ for the second form. The mean proportional of two positive numbers a and b is $\sqrt{ab}$; for $a \neq 0$, the third proportional to a and b is $\frac{b^2}{a}$ and the fourth proportional to a, b and c is $\frac{bc}{a}$ (so that $a : b = c : \frac{bc}{a}$).

PERCENTAGE CHANGE

$$\text{percentage change} = \frac{\text{new} - \text{original}}{\text{original}} \times 100$$

Positive for an increase, negative for a decrease. 'A is $x\%$ less than B' uses B as the base; the matching statement 'B is $y\%$ more than A' uses A, so $y = \frac{100x}{100-x}$ for $0 < x < 100$, which is larger than x.

TWO SUCCESSIVE PERCENTAGE CHANGES

$$\text{net change} = \left(a + b + \frac{ab}{100}\right)\%$$

Each change is taken on the value left by the previous step; write an increase as positive and a decrease as negative. For three or more changes multiply the factors: $\left(1+\frac{a}{100}\right)\left(1+\frac{b}{100}\right)\left(1+\frac{c}{100}\right)$.

KEEPING EXPENDITURE CONSTANT

$$\text{price up by } r\% \;\Rightarrow\; \text{consumption down by } \frac{r}{100+r}\times 100\%$$

Expenditure = price × consumption. If instead the price falls by $r\%$ ($r < 100$), consumption can rise by $\frac{r}{100-r}\times 100\%$ for the same expenditure. The same algebra covers any product held fixed (speed × time for a fixed distance).

PROFIT, LOSS AND DISCOUNT

$$\text{profit }\% = \frac{SP - CP}{CP}\times 100, \qquad SP = CP\left(1+\frac{p}{100}\right), \qquad SP = MP\left(1-\frac{d}{100}\right)$$

p is the profit percentage on CP (negative for a loss); d is the discount percentage on MP. Successive discounts combine like successive changes, so 20% and 10% make a single discount of 28%, not 30%.

SIMPLE INTEREST

$$SI = \frac{P\,R\,T}{100}, \qquad A = P + SI$$

P principal, R rate in per cent per annum, T time in years. Interest is earned on the principal only, so the amount grows by the same sum every year (an AP).

COMPOUND INTEREST

$$A = P\left(1+\frac{R}{100}\right)^{T}, \qquad A = P\left(1+\frac{R}{100k}\right)^{kT} \text{ when compounded } k \text{ times a year}$$

Interest is added to the principal at the end of each period, so the amount grows by a constant factor (a GP). $CI = A - P$. For different rates in successive years multiply the factors $\left(1+\frac{R_1}{100}\right)\left(1+\frac{R_2}{100}\right)\cdots$. The same formula gives population growth at $R\%$ a year, and a value that depreciates by $R\%$ a year becomes $P\left(1-\frac{R}{100}\right)^{T}$.

DIFFERENCE BETWEEN COMPOUND AND SIMPLE INTEREST

$$CI - SI = P\left(\frac{R}{100}\right)^{2} \ (2 \text{ years}), \qquad CI - SI = P\left(\frac{R}{100}\right)^{2}\left(3+\frac{R}{100}\right) \ (3 \text{ years})$$

Same P and R, compounded annually.

AVERAGE AND WEIGHTED AVERAGE

$$\bar{x} = \frac{x_1 + x_2 + \cdots + x_n}{n}, \qquad \bar{x}_w = \frac{w_1x_1 + w_2x_2 + \cdots + w_nx_n}{w_1 + w_2 + \cdots + w_n}$$

Adding k to every value adds k to the average. Replacing one value changes the sum by (new − old), so the average changes by $\frac{\text{new}-\text{old}}{n}$. Work with sums, not averages, when items join or leave a group.

ALLIGATION (RULE OF MIXTURE)

$$\frac{q_1}{q_2} = \frac{c_2 - m}{m - c_1}, \qquad c_1 < m < c_2$$

Quantity q_1 at price (or concentration) c_1 mixed with q_2 at c_2 gives the mean m. It is the weighted-average formula rearranged, so it works for prices, percentages of a component and marks. It also works for average speeds, provided the quantities are the TIMES spent at each speed, not the distances. Water counts as price 0.

REPEATED REMOVAL AND REPLACEMENT

$$\text{liquid A left} = V\left(1 - \frac{x}{V}\right)^{n}$$

A container is full with V litres of pure liquid A. x litres are drawn off and replaced by liquid B, and this is done n times, with complete mixing each time. Each operation multiplies the amount of A by $\left(1 - \frac{x}{V}\right)$, so if the container instead starts with Q_0 litres of A in a mixture of total volume V, the amount left is $Q_0\left(1 - \frac{x}{V}\right)^{n}$.

SPEED, DISTANCE AND TIME

$$\text{distance} = \text{speed} \times \text{time}, \qquad 1\ \text{km/h} = \frac{5}{18}\ \text{m/s}, \qquad 1\ \text{m/s} = \frac{18}{5}\ \text{km/h}$$

Put every quantity in one unit system before calculating. For a fixed distance, time is inversely proportional to speed.

AVERAGE SPEED

$$v_{\text{avg}} = \frac{\text{total distance}}{\text{total time}}; \qquad \text{equal distances: } \frac{2uv}{u+v}; \qquad \text{equal times: } \frac{u+v}{2}$$

Over equal distances at speeds u and v the average speed is their harmonic mean, which is less than the arithmetic mean unless $u = v$.

RELATIVE SPEED AND TRAINS

$$v_{\text{rel}} = |u - v|\ \text{(same direction)}, \quad v_{\text{rel}} = u + v\ \text{(opposite directions)}; \qquad t = \frac{L_1 + L_2}{v_{\text{rel}}}$$

Time to close a gap or to pass = (length to be covered) ÷ (relative speed). A train of length L passes a pole or a standing person in $\frac{L}{v}$ and a platform of length P in $\frac{L+P}{v}$; two trains pass each other in $\frac{L_1+L_2}{v_{\text{rel}}}$.

BOATS AND STREAMS

$$d = b + s, \quad u = b - s; \qquad b = \frac{d+u}{2}, \quad s = \frac{d-u}{2}$$

b is the speed in still water, s the speed of the stream, d the downstream and u the upstream speed ($b > s$ for the boat to make headway upstream).

TIME AND WORK; PIPES AND CISTERNS

$$\text{rate} = \frac{1}{\text{time to finish alone}}, \qquad \frac{1}{T} = \frac{1}{a} + \frac{1}{b} \Rightarrow T = \frac{ab}{a+b}$$

Rates add when workers act together; an outlet pipe, or anyone undoing the job, has a negative rate. Taking the total work as the LCM of the given times keeps every rate a whole number of units per day (or hour).

LAWS OF EXPONENTS

$$a^m a^n = a^{m+n}, \quad \frac{a^m}{a^n} = a^{m-n}, \quad (a^m)^n = a^{mn}, \quad (ab)^n = a^n b^n, \quad \left(\frac{a}{b}\right)^n = \frac{a^n}{b^n}$$

Valid for all real exponents when $a, b > 0$; for integer exponents any non-zero a, b will do. A tower is read from the top: $a^{m^n} = a^{(m^n)}$, which is not $(a^m)^n$.

ZERO, NEGATIVE AND FRACTIONAL EXPONENTS

$$a^0 = 1\ (a \neq 0), \qquad a^{-n} = \frac{1}{a^n}\ (a \neq 0), \qquad a^{p/q} = \left(\sqrt[q]{a}\right)^p = \sqrt[q]{a^p}\ (a > 0)$$

For $a > 0$, $\sqrt[q]{a}$ is the positive q-th root.

SURDS AND RATIONALISING

$$\sqrt{a}\,\sqrt{b} = \sqrt{ab}, \qquad \frac{1}{\sqrt{a} + \sqrt{b}} = \frac{\sqrt{a} - \sqrt{b}}{a - b}$$

For $a, b \geq 0$, with $a \neq b$ in the second. Multiply by the conjugate to clear a root from the denominator. $\sqrt{a+b} \neq \sqrt{a} + \sqrt{b}$.

DEFINITION OF A LOGARITHM

$$\log_a x = y \;\Leftrightarrow\; a^y = x \qquad (a > 0,\ a \neq 1,\ x > 0)$$

The logarithm exists only for a positive argument and a positive base other than 1. $\log_a 1 = 0$ and $\log_a a = 1$ for every valid base. For $a > 1$, $\log_a x$ increases as x increases; for $0 < a < 1$ it decreases, so taking logarithms to such a base reverses an inequality.

LAWS OF LOGARITHMS

$$\log_a(xy) = \log_a x + \log_a y, \qquad \log_a \frac{x}{y} = \log_a x - \log_a y, \qquad \log_a x^n = n \log_a x$$

For $x, y > 0$. There is no law for $\log_a(x + y)$.

CHANGE OF BASE

$$\log_a x = \frac{\log_b x}{\log_b a}, \qquad \log_a b \cdot \log_b a = 1, \qquad \log_{a^k} x = \frac{1}{k} \log_a x$$

Any valid base b may be used; choose 10, e, or whichever base makes the numbers simplest. The last form needs $k \neq 0$.

TWO IDENTITIES

$$a^{\log_a x} = x, \qquad x^{\log_a y} = y^{\log_a x}$$

For $x, y > 0$. To verify the second, take $\log_a$ of both sides: each becomes $\log_a x \cdot \log_a y$.

DIGITS OF A LARGE NUMBER

$$\text{number of digits of } N = \lfloor \log_{10} N \rfloor + 1 \quad (N \text{ a positive integer})$$

$\lfloor \cdot \rfloor$ is the greatest integer not exceeding the value. For a non-integer $N \geq 1$ the same expression counts the digits before the decimal point. For $0 < N < 1$, the number of zeros between the decimal point and the first non-zero digit is $\lceil -\log_{10} N \rceil - 1$.

DIVISION ALGORITHM

$$a = bq + r, \qquad 0 \leq r < b$$

For an integer a and a positive integer b, the quotient q and remainder r are unique. A reported remainder is never negative: a 'remainder of -1' on division by 7 is only a working shortcut and means 6.

REMAINDERS OF SUMS AND PRODUCTS

$$(a + b) \bmod m = \big((a \bmod m) + (b \bmod m)\big) \bmod m, \qquad (ab) \bmod m = \big((a \bmod m)(b \bmod m)\big) \bmod m$$

Replace each number by its remainder before adding or multiplying; remainders of powers follow by repeated multiplication. $a \equiv b \pmod m$ means a and b leave the same remainder on division by m.

FERMAT'S LITTLE THEOREM

$$a^{p-1} \equiv 1 \pmod p$$

For a prime p that does not divide a. For the remainder of a large power on division by p, reduce the exponent modulo $p - 1$.

HCF AND LCM OF TWO NUMBERS

$$\mathrm{HCF}(a, b) \times \mathrm{LCM}(a, b) = a \times b$$

True for exactly two positive integers, not for three or more. The HCF takes the lowest power of each common prime; the LCM takes the highest power of every prime present.

NUMBER AND SUM OF FACTORS

$$N = p_1^{a_1} p_2^{a_2} \cdots p_k^{a_k}: \quad \tau(N) = (a_1+1)(a_2+1)\cdots(a_k+1), \quad \sigma(N) = \prod_{i=1}^{k} \frac{p_i^{a_i+1}-1}{p_i-1}$$

$p_1, \ldots, p_k$ are distinct primes. $\tau(N)$ counts all positive factors including 1 and N; $\sigma(N)$ is their sum. N is a perfect square exactly when $\tau(N)$ is odd. N can be written as a product of two positive factors (order ignored) in $\frac{\tau(N)}{2}$ ways, or $\frac{\tau(N)+1}{2}$ ways when N is a perfect square. Example: $720 = 2^4 \cdot 3^2 \cdot 5$ has $5 \times 3 \times 2 = 30$ factors, so it is a product of two factors in 15 ways.

HIGHEST POWER OF A PRIME IN A FACTORIAL

$$\text{power of prime } p \text{ in } n! = \left\lfloor \frac{n}{p} \right\rfloor + \left\lfloor \frac{n}{p^2} \right\rfloor + \left\lfloor \frac{n}{p^3} \right\rfloor + \cdots$$

Stop when $p^k > n$. The number of trailing zeros of $n!$ is this sum with $p = 5$, since there are always at least as many 2s as 5s. Example: $100!$ ends in $20 + 4 = 24$ zeros.

ARITHMETIC PROGRESSION

$$t_n = a + (n-1)d, \qquad S_n = \frac{n}{2}\left[2a + (n-1)d\right] = \frac{n}{2}(a + l)$$

a first term, d common difference, l last term. The number of terms from a to l is $\frac{l-a}{d} + 1$. Take three terms in AP as $a - d,\ a,\ a + d$ so that their sum is simply $3a$. For any sequence, not only an AP, $t_1 = S_1$ and $t_n = S_n - S_{n-1}$ for $n \geq 2$.

GEOMETRIC PROGRESSION

$$t_n = ar^{n-1}, \qquad S_n = \frac{a\,(r^n - 1)}{r - 1}\ (r \neq 1), \qquad S_\infty = \frac{a}{1 - r}\ (|r| < 1)$$

If $r = 1$, $S_n = na$. An infinite GP has a finite sum only when $|r| < 1$. Take three terms in GP as $\frac{a}{r},\ a,\ ar$ so that their product is a^3.

MEANS OF TWO POSITIVE NUMBERS

$$AM = \frac{a+b}{2}, \quad GM = \sqrt{ab}, \quad HM = \frac{2ab}{a+b}; \qquad AM \geq GM \geq HM, \quad GM^2 = AM \times HM$$

Equality holds only when $a = b$. A harmonic progression is a sequence whose reciprocals form an AP. AM ≥ GM also holds for n positive numbers: with the sum fixed, the product is greatest when the numbers are equal, and with the product fixed, the sum is least when they are equal.

SUMS OF POWERS OF THE FIRST N NATURAL NUMBERS

$$\sum_{k=1}^{n} k = \frac{n(n+1)}{2}, \qquad \sum_{k=1}^{n} k^2 = \frac{n(n+1)(2n+1)}{6}, \qquad \sum_{k=1}^{n} k^3 = \left[\frac{n(n+1)}{2}\right]^2$$

Also: the sum of the first n odd numbers is n^2 and the sum of the first n even numbers is $n(n+1)$.

TELESCOPING BY PARTIAL FRACTIONS

$$\frac{1}{k(k+d)} = \frac{1}{d}\left(\frac{1}{k} - \frac{1}{k+d}\right) \Rightarrow \sum_{k=1}^{n} \frac{1}{k(k+1)} = 1 - \frac{1}{n+1} = \frac{n}{n+1}$$

Split each term into a difference so that neighbouring pieces cancel; only the first and last pieces survive.

ARRANGEMENTS AND SELECTIONS OF DISTINCT OBJECTS

$$^nP_r = \frac{n!}{(n-r)!}, \qquad ^nC_r = \frac{n!}{r!\,(n-r)!} = \frac{^nP_r}{r!}$$

$0 \leq r \leq n$, no repetition. $^nP_n = n!$, $^nC_0 = {}^nC_n = 1$ and $0! = 1$.

PROPERTIES OF NCR

$$ {}^{n}C_{r} = {}^{n}C_{n-r}, \qquad {}^{n}C_{r} + {}^{n}C_{r-1} = {}^{n+1}C_{r}, \qquad \sum_{r=0}^{n} {}^{n}C_{r} = 2^{n} $$

If ${}^{n}C_{x} = {}^{n}C_{y}$ then $x = y$ or $x + y = n$. The number of ways to select at least one item from n distinct items is $2^{n} - 1$.

BINOMIAL THEOREM

$$ (x+y)^{n} = \sum_{r=0}^{n} {}^{n}C_{r}\, x^{n-r} y^{r}, \qquad T_{r+1} = {}^{n}C_{r}\, x^{n-r} y^{r} $$

For a positive integer n. There are $n+1$ terms and T_{r+1} is the general term. Putting $x = y = 1$ gives $\sum_{r=0}^{n} {}^{n}C_{r} = 2^{n}$.

INCLUSION AND EXCLUSION

$$ n(A \cup B) = n(A) + n(B) - n(A \cap B), \qquad n(A \cup B \cup C) = n(A) + n(B) + n(C) - n(A \cap B) - n(B \cap C) - n(C \cap A) + n(A \cap B \cap C) $$

Anything counted twice is subtracted once, and so on. Among $1, 2, \ldots, N$ exactly $\left\lfloor \frac{N}{k} \right\rfloor$ numbers are multiples of k. Example: from 1 to 100, the numbers divisible by 2 or 3 number $50 + 33 - 16 = 67$ (16 being the multiples of 6).

REPETITION ALLOWED

$$ n^{r} \text{ arrangements}, \qquad {}^{n+r-1}C_{r} \text{ selections} $$

r positions each filled from n distinct kinds in unlimited supply give n^{r} ordered arrangements. When only how many of each kind are chosen matters, the number of selections of r items from n kinds is ${}^{n+r-1}C_{r}$.

ARRANGEMENTS WITH IDENTICAL OBJECTS

$$ \frac{n!}{p_1!\, p_2! \cdots p_k!} $$

n objects of which p_1 are alike of one kind, p_2 alike of another, and so on ($p_1 + p_2 + \cdots + p_k = n$; an object with no duplicate counts as a kind with $p = 1$). Selecting at least one item from p identical items of one kind, q of a second kind and t of a third can be done in $(p+1)(q+1)(t+1) - 1$ ways.

CIRCULAR ARRANGEMENTS

$$ (n-1)! \text{ around a circle}, \qquad \frac{(n-1)!}{2} \text{ if clockwise and anticlockwise orders count as the same} $$

For n distinct objects only the relative order counts. The halved count ($n \geq 3$) applies to necklaces, garlands and rings of beads that can be turned over, never to people seated round a table. If the seats are numbered, or one position is otherwise marked, the arrangement is linear: $n!$.

DISTRIBUTING N OBJECTS INTO R DISTINCT BOXES

$$ \text{identical objects: } {}^{n+r-1}C_{r-1} \text{ (boxes may be empty)}, \quad {}^{n-1}C_{r-1} \text{ (no box empty)}; \qquad \text{distinct objects: } r^{n} $$

Equivalently, $x_1 + x_2 + \cdots + x_r = n$ has ${}^{n+r-1}C_{r-1}$ solutions in non-negative integers and ${}^{n-1}C_{r-1}$ in positive integers (needs $n \geq r$). The count r^{n} for distinct objects allows empty boxes; with no box empty it is $r^{n} - {}^{r}C_{1}(r-1)^{n} + {}^{r}C_{2}(r-2)^{n} - \cdots$, by inclusion and exclusion. Example: 4 distinct objects into 3 boxes with none empty: $81 - 3 \times 16 + 3 \times 1 = 36$.

COUNTING IN GEOMETRY AND ON GRIDS

$$ \text{lines} = {}^{n}C_{2}, \quad \text{triangles} = {}^{n}C_{3}, \quad \text{diagonals of an } n\text{-gon} = \frac{n(n-3)}{2}; \qquad \text{grid paths} = \frac{(m+n)!}{m!\, n!} $$

Lines and triangles are for n points with no three collinear. If m of them ($m \geq 3$) lie on one line and no other three are collinear, subtract ${}^{m}C_{2} - 1$ lines and ${}^{m}C_{3}$ triangles. Grid paths counts the shortest routes made of m unit steps right and n unit steps up in any order.

FIRST-ORDER APPROXIMATION

$$ (1+x)^{n} \approx 1 + nx \quad (|x| \ll 1) $$

Good when $|nx|$ is a few per cent; the neglected term is about $\frac{n(n-1)}{2}x^{2}$. Example: $1.02^{5} \approx 1.10$ (exact value 1.1041).

SQUARE ROOT NEAR A PERFECT SQUARE

$$\sqrt{a^2+b} \approx a + \frac{b}{2a} \quad (|b| \ll a^2)$$

Example: $\sqrt{1604} \approx 40 + \frac{4}{80} = 40.05$; indeed $40.05^2 = 1604.0025$.

RELATIVE CHANGE OF A PRODUCT, QUOTIENT OR POWER

$$\frac{\Delta(xy)}{xy} \approx \frac{\Delta x}{x} + \frac{\Delta y}{y}, \qquad \frac{\Delta(x/y)}{x/y} \approx \frac{\Delta x}{x} - \frac{\Delta y}{y}, \qquad \frac{\Delta\left(x^n\right)}{x^n} \approx n\,\frac{\Delta x}{x}$$

For small relative changes. A 2% rise in length and a 3% rise in width raise the area by about 5% (exactly 5.06%).

Geometry and mensuration

INTERIOR ANGLE SUM OF A POLYGON

$$S_{\text{int}} = (n-2) \times 180^\circ$$

Any simple polygon with n sides (sides do not cross), convex or not: 180° for a triangle, 360° for a quadrilateral, 540° for a pentagon.

EXTERIOR ANGLE SUM OF A CONVEX POLYGON

$$S_{\text{ext}} = 360^\circ$$

One exterior angle taken at each vertex of a convex polygon; the sum is 360° whatever the value of n. At every vertex, interior angle $+$ exterior angle $= 180^\circ$.

EACH ANGLE OF A REGULAR POLYGON

$$\text{exterior} = \frac{360^\circ}{n}$$

$$\text{interior} = 180^\circ - \frac{360^\circ}{n} = \frac{(n-2)\times 180^\circ}{n}$$

Regular polygon only (all sides equal and all angles equal).

NUMBER OF DIAGONALS

$$D = \frac{n(n-3)}{2}$$

A diagonal joins two vertices that are not neighbours. Each vertex joins $n-3$ others by a diagonal (not itself and not its two neighbours); dividing by 2 stops each diagonal being counted twice.

PYTHAGORAS' THEOREM AND THE ANGLE TEST

$$c^2 = a^2 + b^2 \iff \angle C = 90^\circ$$

$$c^2 < a^2 + b^2 \iff \angle C < 90^\circ$$

$$c^2 > a^2 + b^2 \iff \angle C > 90^\circ$$

c is the side opposite angle C. Apply the test to the longest side: the triangle is acute, right or obtuse according to the angle opposite that side.

AREA OF A TRIANGLE

$$\Delta = \frac{1}{2}\,bh = \frac{1}{2}\,ab\sin C$$

h is the perpendicular height onto the base b (in an obtuse triangle it can fall outside the triangle, on the extended base); C is the angle between sides a and b.

HERON'S FORMULA

$$\Delta = \sqrt{s(s-a)(s-b)(s-c)}, \qquad s = \frac{a+b+c}{2}$$

Any triangle whose three sides are known. If any of $s-a$, $s-b$, $s-c$ is zero or negative, the three lengths do not form a triangle.

EQUILATERAL TRIANGLE OF SIDE A

$$h = \frac{\sqrt{3}}{2}\,a, \qquad \Delta = \frac{\sqrt{3}}{4}\,a^2$$
$$R = \frac{a}{\sqrt{3}}, \qquad r = \frac{a}{2\sqrt{3}}$$

All four centres coincide at the centroid, which divides each height in the ratio $2:1$ from the vertex. So $R = \frac{2}{3}h$ and $r = \frac{1}{3}h$, giving $R = 2r$ and $h = 3r = \frac{3}{2}R$.

INRADIUS AND CIRCUMRADIUS

$$\Delta = r\,s, \qquad \Delta = \frac{abc}{4R}$$

Any triangle: r is the radius of the incircle (touches all three sides), R the radius of the circumcircle (passes through all three vertices) and s the semi-perimeter.

RIGHT TRIANGLE WITH LEGS A, B AND HYPOTENUSE C

$$R = \frac{c}{2}, \qquad r = \frac{a+b-c}{2}, \qquad h_c = \frac{ab}{c}$$

h_c is the altitude from the right angle to the hypotenuse. If it cuts the hypotenuse into parts p (next to leg a) and q (next to leg b), then $h_c^2 = pq$, $a^2 = pc$ and $b^2 = qc$.

ANGLE BISECTOR THEOREM

$$\frac{BD}{DC} = \frac{AB}{AC}, \qquad AD^2 = AB \cdot AC - BD \cdot DC$$

AD is the internal bisector of angle A, meeting BC at D: it divides the opposite side in the ratio of the two sides that form the angle, and the second relation gives its length. When $AB \neq AC$, the external bisector of angle A meets BC produced at a point E with $BE/EC = AB/AC$.

QUADRILATERAL AREAS

$$\text{Parallelogram} = bh = ab\sin\theta$$
$$\text{Rhombus} = \tfrac{1}{2}\,d_1 d_2$$
$$\text{Trapezium} = \tfrac{1}{2}(a+b)h$$
$$\text{Any quadrilateral} = \tfrac{1}{2}\,d_1 d_2 \sin\theta$$

Parallelogram: θ is the angle between sides a and b, and h the height onto base b. Trapezium: a and b are the parallel sides and h the distance between them. Any quadrilateral: θ is the angle between the diagonals (90° for a rhombus, a square or a kite).

PATHS OF UNIFORM WIDTH

$$\text{outside: } (l+2w)(b+2w) - lb = 2w(l+b+2w)$$
$$\text{inside: } lb - (l-2w)(b-2w) = 2w(l+b-2w)$$
$$\text{two crossing roads: } w(l+b-w)$$

A path of width w running all round the outside or the inside of a rectangular plot $l \times b$. The crossing roads are two roads of width w through the plot, one parallel to each pair of sides; the square where they cross is counted once, which is why w^2 is subtracted. A circular path of width w round the outside of a circle of radius r has area $\pi\left[(r+w)^2 - r^2\right] = \pi w(2r+w)$.

CIRCUMFERENCE AND AREA OF A CIRCLE

$$C = 2\pi r = \pi d, \qquad A = \pi r^2$$

Radius r, diameter d. A wheel of radius r rolling without slipping moves forward $2\pi r$ in one complete revolution.

ARC LENGTH AND SECTOR AREA

$$l = \frac{\theta}{360^\circ} \times 2\pi r$$

$$A_{\text{sector}} = \frac{\theta}{360^\circ} \times \pi r^2 = \frac{1}{2}\, l\, r$$

θ is the angle at the centre, in degrees. With θ in radians the same results read $l = r\theta$ and $A = \frac{1}{2}r^2\theta$.

SEGMENT AREA AND CHORD LENGTH

$$A_{\text{segment}} = r^2 \left(\frac{\pi\theta}{360^\circ} - \frac{1}{2}\sin\theta \right)$$

$$\text{chord} = 2r \sin\frac{\theta}{2} = 2\sqrt{r^2 - d^2}$$

Minor segment cut off by a chord that subtends θ (degrees) at the centre: sector minus triangle. d is the perpendicular distance of the chord from the centre. Major segment $= \pi r^2 -$ minor segment.

TANGENT LENGTHS

$$PT = \sqrt{d^2 - r^2}$$

$$L_{\text{direct}} = \sqrt{d^2 - (r_1 - r_2)^2}$$

$$L_{\text{transverse}} = \sqrt{d^2 - (r_1 + r_2)^2}$$

PT: tangent from a point at distance d from the centre of a circle of radius r. For two circles of radii r_1, r_2 with centres d apart: a direct common tangent has both circles on the same side of it and exists only if $d \geq |r_1 - r_2|$; a transverse common tangent passes between them and exists only if $d \geq r_1 + r_2$. Each length is measured between the two points of contact.

DISTANCE AND MID-POINT

$$d = \sqrt{(x_2 - x_1)^2 + (y_2 - y_1)^2}, \qquad M = \left(\frac{x_1 + x_2}{2}, \frac{y_1 + y_2}{2} \right)$$

Between $A(x_1, y_1)$ and $B(x_2, y_2)$. The distance of (x, y) from the origin is $\sqrt{x^2 + y^2}$.

SECTION FORMULA

$$\text{internal: } P = \left(\frac{mx_2 + nx_1}{m + n}, \frac{my_2 + ny_1}{m + n} \right)$$

$$\text{external: } P = \left(\frac{mx_2 - nx_1}{m - n}, \frac{my_2 - ny_1}{m - n} \right)$$

P divides AB in the ratio $m : n$, that is $AP : PB = m : n$, from $A(x_1, y_1)$ towards $B(x_2, y_2)$. External division needs $m \neq n$. The ratio $1 : 1$ gives the mid-point.

CENTROID AND INCENTRE FROM COORDINATES

$$G = \left(\frac{x_1 + x_2 + x_3}{3}, \frac{y_1 + y_2 + y_3}{3} \right)$$

$$I = \left(\frac{ax_1 + bx_2 + cx_3}{a + b + c}, \frac{ay_1 + by_2 + cy_3}{a + b + c} \right)$$

Triangle $A(x_1, y_1)$, $B(x_2, y_2)$, $C(x_3, y_3)$ with $a = BC$, $b = CA$, $c = AB$: in the incentre, each vertex is weighted by the length of the side opposite it.

AREA OF A TRIANGLE FROM ITS VERTICES

$$\Delta = \frac{1}{2}\left| x_1(y_2 - y_3) + x_2(y_3 - y_1) + x_3(y_1 - y_2) \right|$$

Take the absolute value: the expression inside is negative when the vertices are listed clockwise. $\Delta = 0$ means the three points are collinear. For a polygon, list the vertices in order round the boundary and use the shoelace sum $\frac{1}{2}\left|\sum(x_i y_{i+1} - x_{i+1} y_i)\right|$, returning to the first vertex at the end.

SLOPE, PARALLEL AND PERPENDICULAR LINES

$$m = \tan\theta = \frac{y_2 - y_1}{x_2 - x_1} \quad (x_1 \neq x_2)$$

$$\text{parallel: } m_1 = m_2, \qquad \text{perpendicular: } m_1 m_2 = -1$$

θ is the angle the line makes with the positive x-axis, measured anticlockwise. A horizontal line has $m = 0$; a vertical line has no defined slope. The perpendicularity condition applies when neither line is vertical (a horizontal line is perpendicular to a vertical one).

EQUATIONS OF A STRAIGHT LINE

$y = mx + c$	slope m, y-intercept c
$y - y_1 = m(x - x_1)$	slope m, through (x_1, y_1)
$y - y_1 = \frac{y_2 - y_1}{x_2 - x_1}(x - x_1)$	through (x_1, y_1) and (x_2, y_2)
$\frac{x}{a} + \frac{y}{b} = 1$	x-intercept a, y-intercept b
$Ax + By + C = 0$	slope $-\frac{A}{B}$ $(B \neq 0)$

The two-point form needs $x_1 \neq x_2$ (otherwise the line is $x = x_1$). The intercept form needs $a \neq 0$ and $b \neq 0$, so it cannot describe a line through the origin or a line parallel to an axis. A horizontal line is $y = k$ and a vertical line is $x = k$. The line $\frac{x}{a} + \frac{y}{b} = 1$ forms a right triangle of area $\frac{1}{2}|ab|$ with the coordinate axes.

DISTANCE FROM A POINT TO A LINE

$$p = \frac{|Ax_1 + By_1 + C|}{\sqrt{A^2 + B^2}}$$

$$\text{parallel lines: } \frac{|C_1 - C_2|}{\sqrt{A^2 + B^2}}$$

Perpendicular distance of (x_1, y_1) from $Ax + By + C = 0$. For the parallel lines $Ax + By + C_1 = 0$ and $Ax + By + C_2 = 0$, first make the x and y coefficients identical in both equations.

CUBE AND CUBOID

$$\text{Cube: } V = a^3, \quad \text{TSA} = 6a^2, \quad \text{diagonal} = a\sqrt{3}$$

$$\text{Cuboid: } V = lbh, \quad \text{TSA} = 2(lb + bh + hl)$$

$$\text{diagonal} = \sqrt{l^2 + b^2 + h^2}$$

The diagonal here is the space (body) diagonal, from a corner to the opposite corner through the interior; a face diagonal of a cube is $a\sqrt{2}$. The four walls of a room of length l, breadth b and height h have area $2h(l + b)$.

RIGHT CIRCULAR CYLINDER

$$V = \pi r^2 h, \qquad \text{CSA} = 2\pi r h, \qquad \text{TSA} = 2\pi r(r + h)$$

Hollow cylinder (pipe) of outer radius R, inner radius r and length h: volume of material $= \pi h(R^2 - r^2)$.

RIGHT CIRCULAR CONE

$$l = \sqrt{r^2 + h^2}, \qquad V = \frac{1}{3}\pi r^2 h$$

$$\text{CSA} = \pi r l, \qquad \text{TSA} = \pi r(r + l)$$

h is the vertical height and l the slant height; do not interchange them. A cone has one third of the volume of a cylinder with the same base and height.

SPHERE AND HEMISPHERE

$$\text{Sphere: } V = \tfrac{4}{3}\pi r^3, \quad S = 4\pi r^2$$
$$\text{Hemisphere: } V = \tfrac{2}{3}\pi r^3, \quad \text{CSA} = 2\pi r^2, \quad \text{TSA} = 3\pi r^2$$

A solid hemisphere's total surface includes its flat circular face (πr^2). Spherical shell with outer radius R and inner radius r: volume of material $= \tfrac{4}{3}\pi(R^3 - r^3)$.

FRUSTUM OF A CONE

$$V = \frac{\pi h}{3}\left(R^2 + Rr + r^2\right), \qquad l = \sqrt{h^2 + (R-r)^2}$$
$$\text{CSA} = \pi(R + r)\, l$$
$$\text{TSA} = \pi(R + r)\, l + \pi R^2 + \pi r^2$$

The part of a cone between its base and a cut parallel to the base; R and r are the radii of the two circular ends and h is the perpendicular distance between them. Putting $r = 0$ gives the cone formulas.

SCALING LAW FOR SIMILAR FIGURES

$$\frac{L_2}{L_1} = k, \qquad \frac{A_2}{A_1} = k^2, \qquad \frac{V_2}{V_1} = k^3$$

Holds only when ALL linear dimensions change by the same factor. Doubling the radius of a sphere multiplies its surface area by 4 and its volume by 8.

PERCENTAGE CHANGE UNDER UNIFORM SCALING

$$\text{new area} = \left(1 + \frac{x}{100}\right)^2 \times \text{old area}$$
$$\text{new volume} = \left(1 + \frac{x}{100}\right)^3 \times \text{old volume}$$

Every length changes by $x\%$ of its original value ($x < 0$ for a decrease). A 10% increase in every length raises area by 21% and volume by 33.1%; a 10% decrease lowers area by 19% and volume by 27.1%.

Elementary statistics and probability

ARITHMETIC MEAN

$$\bar{x} = \frac{x_1 + x_2 + \cdots + x_n}{n} = \frac{1}{n}\sum_{i=1}^{n} x_i$$

Uses every observation, so one extreme value can move it a lot. Total of all values $= n\bar{x}$, and many mean questions become simple once they are worked through totals.

WEIGHTED MEAN (ALSO THE MEAN OF A FREQUENCY TABLE)

$$\bar{x}_w = \frac{\sum w_i x_i}{\sum w_i}$$

Weights $w_i \geq 0$, not all zero. For a frequency table put $w_i = f_i$, the frequency of the value x_i. For grouped data (class intervals), take x_i as the mid-point of each class.

COMBINED MEAN OF TWO GROUPS

$$\bar{x} = \frac{n_1\bar{x}_1 + n_2\bar{x}_2}{n_1 + n_2}$$

The group sizes act as weights. The combined mean always lies between the two group means (it equals both when they are equal). When the group sizes differ, it lies nearer the mean of the larger group; when the sizes are equal, it is exactly midway.

EMPIRICAL RELATION BETWEEN THE THREE AVERAGES

$$\text{Mode} \approx 3\,\text{Median} - 2\,\text{Mean}$$

Approximate, and only for moderately skewed data with a single peak. Use it when a question asks for it, or gives two of the three averages and asks for the third. Equivalent form: $\text{Mean} - \text{Mode} \approx 3(\text{Mean} - \text{Median})$.

GEOMETRIC MEAN, HARMONIC MEAN AND THE AM-GM-HM INEQUALITY

$$\text{GM} = (x_1 x_2 \cdots x_n)^{1/n}, \qquad \text{HM} = \frac{n}{\dfrac{1}{x_1} + \dfrac{1}{x_2} + \cdots + \dfrac{1}{x_n}}, \qquad \text{AM} \geq \text{GM} \geq \text{HM}$$

For positive values only. Equality holds throughout exactly when all the values are equal. For two positive numbers a and b: $\text{AM} = \dfrac{a+b}{2}$, $\text{GM} = \sqrt{ab}$, $\text{HM} = \dfrac{2ab}{a+b}$, and $\text{GM}^2 = \text{AM} \times \text{HM}$.

RANGE

$$\text{Range} = x_{\max} - x_{\min}$$

Depends only on the two extreme values, so a single outlier changes it.

VARIANCE (POPULATION)

$$\sigma^2 = \frac{1}{n}\sum_{i=1}^{n}(x_i - \bar{x})^2 = \frac{1}{n}\sum_{i=1}^{n} x_i^2 - \bar{x}^2$$

The mean of the squares minus the square of the mean. Convention in these notes: divide by n unless the question says 'sample'. The variance is always ≥ 0 and is zero only when all values are equal.

SAMPLE VARIANCE

$$s^2 = \frac{1}{n-1}\sum_{i=1}^{n}(x_i - \bar{x})^2$$

Use it only when a question explicitly asks for the sample variance or sample standard deviation ($n \geq 2$).

STANDARD DEVIATION AND COEFFICIENT OF VARIATION

$$\sigma = \sqrt{\sigma^2}, \qquad \text{CV} = \frac{\sigma}{\bar{x}} \times 100\%$$

The SD is in the same units as the data and is never negative. The CV is used for data with a positive mean. It compares the consistency of data sets with different means: the set with the smaller CV is the more consistent.

VARIANCE OF AN ARITHMETIC PROGRESSION

$$\sigma^2 = \frac{d^2(n^2-1)}{12}$$

For n terms in AP with common difference d. For the first n natural numbers this is $\dfrac{n^2-1}{12}$. The mean and the median of an AP both equal $\dfrac{\text{first} + \text{last}}{2}$.

CLASSICAL PROBABILITY

$$P(A) = \frac{n(A)}{n(S)} = \frac{\text{number of outcomes favourable to } A}{\text{total number of outcomes}}$$

Valid only when S is finite and its outcomes are equally likely. Always $0 \leq P(A) \leq 1$, $P(S) = 1$ and $P(\emptyset) = 0$.

COUNTING NOTATION USED IN THESE NOTES

$$^{n}C_r = \frac{n!}{r!\,(n-r)!}, \qquad ^{n}P_r = \frac{n!}{(n-r)!}$$

$^{n}C_r$ counts selections of r items from n distinct items when order does not matter; $^{n}P_r$ counts arrangements, where order matters. Here $0 \leq r \leq n$ and $0! = 1$. When $r > n$ no such selection exists, and $^{n}C_r$ is taken as 0.

COMPLEMENT RULE

$$P(A') = 1 - P(A)$$

A' ('not A') contains every outcome of S that is outside A. Use this rule whenever 'not A' is easier to count than A, above all for 'at least one' questions.

ADDITION RULE (TWO EVENTS)

$$P(A \cup B) = P(A) + P(B) - P(A \cap B)$$

Always true. For mutually exclusive events $P(A \cap B) = 0$, so it reduces to $P(A) + P(B)$.

ADDITION RULE (THREE EVENTS)

$$P(A \cup B \cup C) = P(A) + P(B) + P(C) - P(A \cap B) - P(B \cap C) - P(C \cap A) + P(A \cap B \cap C)$$

Inclusion and exclusion: add the single events, subtract the pairs, then add back the triple.

EXACTLY ONE, AND NEITHER

$$P(\text{exactly one of } A, B) = P(A) + P(B) - 2P(A \cap B), \qquad P(\text{neither}) = 1 - P(A \cup B)$$

'A only' has probability $P(A) - P(A \cap B)$. 'Neither' means not A and not B, that is $P(A' \cap B')$.

CONDITIONAL PROBABILITY

$$P(A \mid B) = \frac{P(A \cap B)}{P(B)}, \quad P(B) > 0$$

With equally likely outcomes this is $P(A \mid B) = \dfrac{n(A \cap B)}{n(B)}$: count inside B only.

MULTIPLICATION RULE

$$P(A \cap B) = P(A)\,P(B \mid A) = P(B)\,P(A \mid B)$$

The first form needs $P(A) > 0$ and the second $P(B) > 0$, so that the conditional probability is defined. For successive draws: $P(A \cap B \cap C) = P(A)\,P(B \mid A)\,P(C \mid A \cap B)$.

INDEPENDENT EVENTS

$$P(A \cap B) = P(A)\,P(B) \iff P(A \mid B) = P(A)$$

The second form needs $P(B) > 0$. If A and B are independent, so are A and B', A' and B, and A' and B'. Successive tosses, rolls, and draws WITH replacement are independent.

AT LEAST ONE

$$P(\text{at least one of } A_1, \ldots, A_n) = 1 - (1 - p_1)\,(1 - p_2) \cdots (1 - p_n)$$

For INDEPENDENT events with $P(A_i) = p_i$. For n independent trials, each with success probability p, it becomes $1 - (1 - p)^n$. 'At least one' covers exactly one, exactly two, and so on up to all of them.

BINOMIAL PROBABILITY

$$P(\text{exactly } k \text{ successes in } n \text{ trials}) = {}^{n}C_{k}\, p^{k} (1 - p)^{n-k}, \quad k = 0, 1, \ldots, n$$

For n independent trials, each with the same success probability p (tosses of a coin, rolls of a die, draws WITH replacement). For a fair coin it is ${}^{n}C_{k} / 2^{n}$.

TOTAL PROBABILITY AND REVERSING A CONDITIONAL PROBABILITY

$$P(A) = \sum_{i=1}^{k} P(B_i)\,P(A \mid B_i), \qquad P(B_j \mid A) = \frac{P(B_j)\,P(A \mid B_j)}{\sum_{i=1}^{k} P(B_i)\,P(A \mid B_i)}$$

$B_1, \ldots, B_k$ must be mutually exclusive and exhaustive (exactly one of them happens), with each $P(B_i) > 0$. The second formula also needs $P(A) > 0$. Use it when a question gives $P(A \mid B)$ but asks for $P(B \mid A)$.

TOTALS OF TWO DICE

$$n(\text{total} = s) = 6 - |s - 7|, \quad s = 2, 3, \ldots, 12$$

Counts ordered pairs, giving 1, 2, 3, 4, 5, 6, 5, 4, 3, 2, 1 ways for totals 2 to 12 (36 in all). Totals s and $14 - s$ are equally likely.

DRAWING WITHOUT REPLACEMENT

$$P(\text{exactly } r \text{ of type 1}) = \frac{{}^{a}C_{r} \times {}^{N-a}C_{k-r}}{{}^{N}C_{k}}$$

For k items drawn together (or one by one without replacement) from N items, of which a are of type 1. The probability is 0 unless $r \le a$ and $k - r \le N - a$. WITH replacement, use the binomial formula with $p = a/N$ instead.

EXPECTED VALUE

$$E(X) = \sum_{i} x_i \, p_i, \qquad \sum_{i} p_i = 1$$

$p_i = P(X = x_i)$, summed over ALL possible values, including zero and negative values such as a loss.

LINEARITY OF EXPECTATION

$$E(aX + b) = a\,E(X) + b, \qquad E(X + Y) = E(X) + E(Y)$$

Holds for any random variables; independence is NOT needed. So the expected total over n repeated games is n times the expected value of one game.

VARIANCE OF A RANDOM VARIABLE

$$\text{Var}(X) = E\left(X^2\right) - [E(X)]^2, \qquad \text{Var}(aX + b) = a^2 \ \text{Var}(X)$$

The same 'mean of the squares minus the square of the mean' idea as for data. For one fair die, $E(X) = \frac{7}{2}$, $E\left(X^2\right) = \frac{91}{6}$ and $\text{Var}(X) = \frac{91}{6} - \frac{49}{4} = \frac{35}{12}$.

TWO STANDARD EXPECTATIONS

$$E(\text{successes in } n \text{ trials}) = np, \qquad E(\text{trials up to and including the first success}) = \frac{1}{p}$$

For independent trials with a constant success probability p ($0 < p \le 1$). The number of successes in n trials also has variance $np(1 - p)$. For example, a fair die takes 6 rolls on average to show its first 6.

Logical and analytical reasoning

CONTRAPOSITIVE

$$(P \to Q) \equiv (\neg Q \to \neg P)$$

Always equivalent to the original. The converse $Q \to P$ and the inverse $\neg P \to \neg Q$ do NOT follow from $P \to Q$ (they are equivalent to each other, not to the original).

IMPLICATION AS 'OR', AND ITS DENIAL

$$(P \to Q) \equiv (\neg P \lor Q), \qquad \neg(P \to Q) \equiv (P \land \neg Q)$$

'If P, then Q' is false only when P is true and Q is false. So the denial of 'If it rains, the match is called off' is 'It rains and the match is not called off', not 'If it rains, the match is not called off'.

IF, ONLY IF, UNLESS

$$Q \text{ if } P \ \equiv \ P \text{ only if } Q \ \equiv \ P \to Q, \qquad P \text{ unless } Q \ \equiv \ \neg Q \to P$$

'Only if' introduces the necessary condition. 'Unless' means 'if not'. 'P if and only if Q' means both $P \to Q$ and $Q \to P$.

NEGATING QUANTIFIED STATEMENTS

$$\neg(\text{All } A \text{ are } B) \equiv \text{Some } A \text{ are not } B, \qquad \neg(\text{Some } A \text{ are } B) \equiv \text{No } A \text{ is } B$$

The negation of 'all' is 'some ... not', not 'none'. Use these to find contradictory pairs and to run the negation test for assumptions.

DE MORGAN'S LAWS

$$\neg(P \wedge Q) \equiv \neg P \vee \neg Q, \qquad \neg(P \vee Q) \equiv \neg P \wedge \neg Q$$

Read 'or' as inclusive (one or both) unless the wording makes it exclusive ('but not both', 'exactly one of').

ARITHMETIC AND GEOMETRIC SERIES

$$a_n = a_1 + (n-1)d, \qquad a_n = a_1 r^{\,n-1}$$

n-th term of an arithmetic series (constant difference d) and of a geometric series (constant ratio r). Positions are counted from $n = 1$.

SERIES WITH CONSTANT SECOND DIFFERENCES

$$a_n = a_1 + (n-1)\Delta_1 + \frac{(n-1)(n-2)}{2}\,c$$

Applies when the second differences are all equal to c; $\Delta_1 = a_2 - a_1$ is the first difference. Example: 5, 11, 23, 41, 65 has $\Delta_1 = 6$ and $c = 6$, so $a_6 = 5 + 5 \times 6 + 10 \times 6 = 95$.

LETTER POSITIONS

$$\mathrm{A} = 1,\ \mathrm{B} = 2,\ \ldots,\ \mathrm{Z} = 26, \qquad \text{opposite of position } n = 27 - n$$

Opposite (mirror) pairs A–Z, B–Y, ..., M–N have positions that add to 27. A forward shift of k with wrap-around gives position $((n + k - 1) \bmod 26) + 1$, so Y shifted forward by 3 is B. A backward shift of k is the same as a forward shift of $26 - k$, so C shifted back by 3 is Z.

RANK IN A ROW

$$N = L + R - 1$$

For one person who is L-th from the left and R-th from the right in a single row of N people. The same holds for ranks from the top and the bottom of a list with no ties.

PEOPLE BETWEEN TWO PERSONS IN A ROW

$$\text{between} = \begin{cases} N - (a + b), & a + b \le N \\ a + b - N - 2, & a + b \ge N + 2 \end{cases}$$

X is a-th from the left and Y is b-th from the right in a row of N. If $a + b = N + 1$, X and Y are the same person. Convert both to positions from the same end if in doubt: Y is at position $N - b + 1$ from the left.

NUMBER OF SEATINGS

$$\text{row: } n!, \qquad \text{circle: } (n-1)!, \qquad \text{circle, mirror images the same: } \frac{(n-1)!}{2}$$

Ways to seat n distinct people. Around a circle only relative positions matter, so rotations count as one arrangement. Divide by 2 only when clockwise and anticlockwise orders are not distinguished, as for a garland of distinct beads ($n \ge 3$).

STRAIGHT-LINE DISTANCE

$$d = \sqrt{(\Delta x)^2 + (\Delta y)^2}, \qquad \Delta x = \text{net distance east}, \quad \Delta y = \text{net distance north}$$

Take east and north as positive, west and south as negative. Name the direction by its quadrant (for example south-east when the net movement is east and south), or by the axis if one component is zero.

SPEEDS OF THE HANDS

$$\text{minute hand: } 6^\circ/\text{min}, \qquad \text{hour hand: } 0.5^\circ/\text{min}, \qquad \text{relative: } 5.5^\circ/\text{min}$$

At H hours M minutes (H on the 12-hour dial, using 0 for 12) the hour hand is $30H + 0.5M$ degrees past 12 and the minute hand is $6M$ degrees past 12, both measured clockwise.

ANGLE BETWEEN THE HANDS

$$\theta = \left|30H - \frac{11}{2}M\right|, \qquad \text{smaller angle} = \min(\theta,\ 360^\circ - \theta)$$

Angle in degrees at H hours M minutes, with H the hour on the 12-hour dial (use 0 for 12). Example: at 7:24, $\theta = |210 - 132| = 78^\circ$.

TIME FOR A GIVEN ANGLE

$$M = \frac{2}{11}(30H \pm \theta) \quad \text{or} \quad M = \frac{2}{11}(30H \pm (360 - \theta))$$

Minutes past H o'clock (H from 0 to 11, using 0 for 12) at which the hands are θ degrees apart, for $0^\circ \le \theta \le 180^\circ$. These values solve $30H - \frac{11}{2}M = \pm\theta$ or $\pm(360^\circ - \theta)$. Keep only values with $0 \le M < 60$. In the first pair, the minus sign gives a moment before the minute hand passes the hour hand and the plus sign a moment after. Try the second pair whenever a value from the first pair falls outside the range: between 1 and 2 o'clock the hands are at right angles at $M = \frac{2}{11}(30 + 90) = 21\frac{9}{11}$ and at $M = \frac{2}{11}(30 + 270) = 54\frac{6}{11}$, and the second time comes only from the second pair.

COINCIDENCES AND COUNTS

$$\text{interval between coincidences} = \frac{720}{11}\ \text{min} = 65\tfrac{5}{11}\ \text{min}$$

For a correct clock. In 12 hours the hands coincide 11 times, are opposite 11 times, and are at right angles 22 times (any angle strictly between 0° and 180° occurs 22 times). In 24 hours each count doubles.

CLOCK SEEN IN A MIRROR

$$\text{mirror time} = (11 - H)\ \text{h}\ (60 - M)\ \text{min} \quad (M \neq 0), \qquad (12 - H)\ \text{h} \quad (M = 0)$$

For an upright plane mirror beside the clock, which swaps left and right. H is the hour on the 12-hour dial (use 0 for 12), and a result of 0 hours is read as 12. It is the same as subtracting the actual time from 12:00: the image of 4:20 reads 7:40, the image of 3:00 reads 9:00 and the image of 12:20 reads 11:40.

LEAP YEAR (GREGORIAN RULE)

$$Y \text{ is a leap year} \iff (4 \mid Y \text{ and } 100 \nmid Y) \text{ or } 400 \mid Y$$

2028 and 2000 are leap years; 2100 and 1900 are not.

ODD DAYS

$$\text{ordinary year: } 1, \quad \text{leap year: } 2, \quad \text{years 1 to 100: } 5, \quad \text{1 to 200: } 3, \quad \text{1 to 300: } 1, \quad \text{1 to 400: } 0$$

The pattern repeats every 400 years, so years 1601 to 1900 also give 1 odd day. Months: 31 days = 3 odd days, 30 days = 2, February = 0 (1 in a leap year).

WEEKDAY FROM ODD DAYS

$$\text{weekday code} = (\text{odd days counted from 1 January of year 1}) \bmod 7, \qquad 0 = \text{Sunday},\ 1 = \text{Monday},\ \ldots,\ 6 = \text{Saturday}$$

Count the complete years before the given year (using the values above), then the complete months of the given year, then the day of the month itself (Gregorian rules applied throughout).

SAME DATE, LATER YEAR

$$\text{shift} \equiv (\text{number of years}) + (\text{number of 29 Februaries in between}) \pmod 7$$

The weekday of a date moves forward by 1 each year, and by 2 when a 29 February falls between the two dates. From Saturday 15 August 2026 to 15 August 2031 the shift is $5 + 1 = 6$ (29 February 2028 lies between), so 15 August 2031 is a Friday.

CHAINS THAT GIVE A DEFINITE RELATION

$$A > B \geq C \Rightarrow A > C, \qquad A \geq B \geq C \Rightarrow A \geq C, \qquad A \geq B,\ B \geq A \Rightarrow A = B$$

A definite relation follows only if every sign on the path between the two terms points the same way; an = sign fits either direction and can sit anywhere in the chain. The relation is = if every sign on the path is =, strict ($>$ or $<$) if at least one sign is strict, and otherwise $\geq$ or $\leq$.

CHAINS THAT GIVE NO RELATION

$$A > B < C \ \Rightarrow\ \text{no definite relation between } A \text{ and } C$$

Signs that point in opposite directions (here B is smaller than both) leave A and C undetermined. The same holds for $A < B > C$ and for $A \geq B \leq C$.

OPERATING ON AN INEQUALITY

$$a > b \Rightarrow a + c > b + c; \qquad a > b,\ c > 0 \Rightarrow ac > bc; \qquad a > b,\ c < 0 \Rightarrow ac < bc; \qquad 0 < a < b \Rightarrow \frac{1}{a} > \frac{1}{b}$$

Multiplying or dividing both sides by a negative number reverses the sign. Squaring is safe only when both sides are non-negative: $-3 < 2$, but $9 > 4$.

TWO GROUPS

$$|A \cup B| = |A| + |B| - |A \cap B|, \qquad \text{neither} = N - |A \cup B|$$

N is the total number of people and $|A|$ the number in group A. The number in A only is $|A| - |A \cap B|$.

THREE GROUPS

$$|A \cup B \cup C| = |A| + |B| + |C| - |A \cap B| - |B \cap C| - |C \cap A| + |A \cap B \cap C|$$

Each pairwise count includes the people who are in all three groups.

EXACTLY ONE, EXACTLY TWO

$$\text{exactly two} = S_2 - 3S_3, \qquad \text{exactly one} = S_1 - 2S_2 + 3S_3$$

Here $S_1 = |A| + |B| + |C|$, $S_2 = |A \cap B| + |B \cap C| + |C \cap A|$ and $S_3 = |A \cap B \cap C|$. Both follow from $S_1 = e_1 + 2e_2 + 3e_3$ and $S_2 = e_2 + 3e_3$, where e_k is the number of people in exactly k of the groups.

LEAST AND GREATEST OVERLAP

$$\max(0,\ |A| + |B| - N) \leq |A \cap B| \leq \min(|A|,\ |B|)$$

For two groups drawn from the same N people, when nothing else is known. If 70% of a class like tea and 60% like coffee, at least 30% and at most 60% of the class like both.

Spatial aptitude

TRANSLATION BY (A, B)

$$(x, y) \to (x + a, y + b)$$

Every point moves by the same amount. Size, shape and orientation are unchanged.

QUARTER AND HALF TURNS ABOUT THE ORIGIN

$$90^\circ \text{ anticlockwise: } (x, y) \to (-y, x)$$
$$90^\circ \text{ clockwise: } (x, y) \to (y, -x)$$
$$180^\circ \text{ either way: } (x, y) \to (-x, -y)$$

These are exact special cases of the rotation rule. For any other centre, apply them to each point's position relative to that centre.

CLOCKWISE AND ANTICLOCKWISE

$$\theta \text{ clockwise} = (360^\circ - \theta) \text{ anticlockwise}$$

So 270° clockwise is 90° anticlockwise. Reduce any angle modulo 360° first: 450° clockwise is 90° clockwise.

REFLECTION IN A MIRROR LINE THROUGH THE ORIGIN

$$x = 0 \text{ (vertical)}: (x, y) \to (-x, y)$$
$$y = 0 \text{ (horizontal)}: (x, y) \to (x, -y)$$
$$y = x: (x, y) \to (y, x)$$
$$y = -x: (x, y) \to (-y, -x)$$

A vertical mirror (the y-axis) swaps left and right. A horizontal mirror (the x-axis) swaps top and bottom. Reflecting twice in the same line gives back the original figure.

SCALING BY A FACTOR K

$$\text{length} \times k, \quad \text{area} \times k^2, \quad \text{volume} \times k^3$$

$k > 1$ enlarges and $0 < k < 1$ shrinks. Angles are unchanged. The same factors apply to similar solids: surface area $\times k^2$, volume $\times k^3$.

CLOCK SEEN IN A VERTICAL MIRROR

$$t_{\text{mirror}} = 11{:}60 - t$$

The mirror stands beside the clock, so it reflects in the line through 12 and 6. Subtract the actual time from 11:60 (that is, from 12:00): 4:25 gives 7:35, and 9:10 gives 2:50. For a time from 12:00 to 12:59, subtract from 23:60 instead (12:10 gives 11:50). Read a result of 0:mm as 12:mm (11:25 gives 0:35, that is 12:35). The rule is exact for both hands.

SMALLEST ANGLE OF ROTATIONAL SYMMETRY

$$\theta_{\min} = \frac{360^\circ}{N}$$

N is the order of rotational symmetry. A regular polygon with n sides has N = n and n lines of symmetry.

HOLES FROM ONE PUNCH THROUGH N COMPLETE HALF-FOLDS

$$\text{holes} = \text{layers under the punch} = 2^n$$

Valid when every fold doubles the whole packet and the punch does not touch any fold line. A punch that touches a fold line gives fewer holes, because holes merge across the fold.

STRIP FOLDED IN HALF N TIMES, THEN CUT ONCE ACROSS

$$\text{pieces} = 2^n + 1$$

All the folds are parallel and each one folds the whole strip in half, so the packet is 2^n layers thick. One straight cut runs right across the packet without touching a fold. It cuts the strip at 2^n places, which leaves $2^n + 1$ pieces. For example, 3 folds and one cut give 9 pieces.

OPPOSITE FACES OF A STANDARD DIE

$$1 + 6 = 2 + 5 = 3 + 4 = 7$$

Use this only when the question says the die is standard or ordinary. For any other die, work out the opposite faces from the views.

PAINTED N × N × N CUBE CUT INTO UNIT CUBES

$$\begin{aligned} 3 \text{ painted faces} &= 8 \\ 2 \text{ painted faces} &= 12(n-2) \\ 1 \text{ painted face} &= 6(n-2)^2 \\ \text{no paint} &= (n-2)^3 \end{aligned}$$

Valid for all six faces painted and $n \geq 2$. The four counts add up to n^3. Cubes with at least one painted face: $n^3 - (n-2)^3 = 6n^2 - 12n + 8$.

PAINTED A × B × C CUBOID CUT INTO UNIT CUBES

$$\begin{aligned} 3 \text{ painted faces} &= 8 \\ 2 \text{ painted faces} &= 4(p+q+r) \\ 1 \text{ painted face} &= 2(pq+qr+rp) \\ \text{no paint} &= pqr \end{aligned}$$

Here p = a − 2, q = b − 2 and r = c − 2. Valid for all six faces painted and a, b, c ≥ 2. If some faces are unpainted, count directly by layer, row and column.

PIECES FROM PLANE CUTS PARALLEL TO THE FACES

$$\text{pieces} = (x+1)(y+1)(z+1)$$

x, y and z are the numbers of cuts in the three directions, with no pieces moved between cuts. For a fixed total number of cuts, the number of pieces is greatest when x, y and z are as nearly equal as possible. To cut a cube into n^3 equal cubes without moving pieces between cuts, the minimum number of plane cuts is $3(n-1)$.

SQUARES IN AN N × N GRID

$$\sum_{k=1}^{n} k^2 = \frac{n(n+1)(2n+1)}{6}$$

All sizes counted. For example, a 3 × 3 grid has 9 + 4 + 1 = 14 squares, and an 8 × 8 chessboard has 204.

SQUARES IN AN M × N GRID (M ≥ N)

$$\sum_{k=1}^{n} (m-k+1)(n-k+1) = \frac{n(n+1)(3m-n+1)}{6}$$

A k × k square fits in (m − k + 1)(n − k + 1) positions, for k = 1 to n. For example, a 4 × 3 grid has 12 + 6 + 2 = 20 squares, and the closed form gives $\frac{3 \times 4 \times 10}{6} = 20$.

RECTANGLES (SQUARES INCLUDED) IN AN M × N GRID

$${}^{m+1}C_2 \times {}^{n+1}C_2 = \frac{m(m+1)n(n+1)}{4}$$

Choose 2 of the m + 1 horizontal lines and 2 of the n + 1 vertical lines. For example, a 4 × 3 grid has 10 × 6 = 60 rectangles, and an 8 × 8 chessboard has 36 × 36 = 1296.

CUBES IN AN N × N × N BLOCK

$$\sum_{k=1}^{n} k^3 = \left(\frac{n(n+1)}{2}\right)^2$$

Cubes of every size are counted. A k × k × k cube fits in $(n-k+1)^3$ positions.

TRIANGLES: K LINES FROM A VERTEX TO THE OPPOSITE SIDE, H LINES PARALLEL TO THAT SIDE

$${}^{k+2}C_2 \times (h+1)$$

The k lines run from the vertex to different interior points of the opposite side, and each parallel line crosses the whole triangle. Every triangle uses two of the k + 2 lines through the vertex and one of the h + 1 lines parallel to the base (the base included).

TRIANGLES IN A TRIANGULAR GRID OF SIDE N

$$\left\lfloor \frac{n(n+2)(2n+1)}{8} \right\rfloor$$

An equilateral triangle is divided into n^2 small equilateral triangles by lines parallel to its sides. The formula counts upright and inverted triangles of all sizes: n = 2 gives 5, n = 3 gives 13 and n = 4 gives 27.

EULER'S FORMULA FOR A CONVEX POLYHEDRON

$$V - E + F = 2$$

V vertices, E edges, F faces. For a cube, 8 − 12 + 6 = 2. A prism on an n-sided base has V = 2n, E = 3n, F = n + 2. A pyramid on an n-sided base has V = n + 1, E = 2n, F = n + 1.

ELEMENT MOVING ROUND N EQUALLY SPACED POSITIONS

$$p_t = (p_0 + st) \bmod n$$

The element starts at position p_0 and moves s positions per step, counted in the stated direction. For an angle, the turn after t steps is $t\theta$ reduced modulo 360°.

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