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Electromagnetics Formula Sheet

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Electrostatics: fields, potential, Gauss's law and capacitance

COULOMB'S LAW

$$\mathbf{F}_2 = \frac{Q_1 Q_2}{4\pi\varepsilon_0 R^2}\,\hat{a}_{12} = \frac{Q_1 Q_2\,(\mathbf{r}_2 - \mathbf{r}_1)}{4\pi\varepsilon_0\,|\mathbf{r}_2 - \mathbf{r}_1|^3}$$

Force on Q_2 due to Q_1; $\hat{a}_{12}$ is the unit vector from Q_1 toward Q_2 and $R = |\mathbf{r}_2 - \mathbf{r}_1|$. A positive product $Q_1 Q_2$ means repulsion. Valid for point charges (or bodies small compared with R). In a homogeneous dielectric filling all space replace ε_0 by ε: the force falls by the factor ε_r.

FIELD OF A POINT CHARGE

$$\mathbf{E} = \frac{\mathbf{F}}{q} = \frac{Q}{4\pi\varepsilon_0 R^2}\,\hat{a}_R$$

Field at distance R from a point charge Q: away from a positive charge, toward a negative one. A charge q placed in the field feels $\mathbf{F} = q\mathbf{E}$. Units: V/m, the same as N/C.

INFINITE LINE CHARGE

$$\mathbf{E} = \frac{\rho_L}{2\pi\varepsilon_0\,\rho}\,\hat{a}_\rho$$

ρ is the perpendicular distance from the line and $\hat{a}_\rho$ points perpendicularly away from it (for $\rho_L > 0$). For a line not on the z-axis, use the perpendicular vector from the line to the field point.

RING OF CHARGE, ON ITS AXIS

$$\mathbf{E} = \frac{Q\,z}{4\pi\varepsilon_0\,(a^2 + z^2)^{3/2}}\,\hat{a}_z$$

Ring of radius a in the plane $z = 0$, centred on the origin, total charge $Q = 2\pi a \rho_L$. Zero at the centre; $|\mathbf{E}|$ is maximum at $z = \pm a/\sqrt{2}$, where it equals $Q/(6\sqrt{3}\,\pi\varepsilon_0 a^2)$; for $|z| \gg a$ it looks like a point charge.

UNIFORMLY CHARGED DISC, ON ITS AXIS

$$\mathbf{E} = \frac{\rho_S}{2\varepsilon_0}\left(1 - \frac{z}{\sqrt{z^2 + a^2}}\right)\hat{a}_z, \qquad z > 0$$

Disc of radius a in the plane $z = 0$. Below the disc ($z < 0$) the field has the same magnitude with $|z|$ in place of z and points along $-\hat{a}_z$. As $a \to \infty$ it becomes the infinite sheet.

INFINITE SHEET OF CHARGE

$$\mathbf{E} = \frac{\rho_S}{2\varepsilon_0}\,\hat{a}_n$$

$\hat{a}_n$ is the unit normal pointing away from the sheet, on each side. The magnitude does not depend on the distance from the sheet.

UNIFORMLY CHARGED SPHERICAL SHELL AND SOLID SPHERE

$$\text{shell: } \mathbf{E} = \begin{cases} 0, & r < a \\ \dfrac{Q}{4\pi\varepsilon_0 r^2}\,\hat{a}_r, & r > a \end{cases} \qquad \text{solid: } \mathbf{E} = \begin{cases} \dfrac{\rho_v\, r}{3\varepsilon_0}\,\hat{a}_r = \dfrac{Q\,r}{4\pi\varepsilon_0 a^3}\,\hat{a}_r, & r \le a \\ \dfrac{Q}{4\pi\varepsilon_0 r^2}\,\hat{a}_r, & r \ge a \end{cases}$$

Radius a, total charge Q (for the solid sphere $Q = \frac{4}{3}\pi a^3 \rho_v$). The shell is a thin surface carrying a uniform ρ_S; an isolated charged conducting sphere, solid or hollow, gives the same field because its charge spreads uniformly over its surface. A shell whose charge is not uniform can have a field inside. Outside, any spherically symmetric charge acts as a point charge at the centre. Inside the solid sphere E grows linearly with r and peaks at the surface.

ELECTRIC FLUX DENSITY

$$\mathbf{D} = \varepsilon_0\mathbf{E} \quad (\text{free space}), \qquad \mathbf{D} = \varepsilon_0\varepsilon_r\mathbf{E} = \varepsilon\mathbf{E} \quad (\text{linear isotropic dielectric})$$

Units C/m². For a point charge $\mathbf{D} = \dfrac{Q}{4\pi r^2}\,\hat{a}_r$, whatever the homogeneous medium around it.

GAUSS'S LAW (INTEGRAL FORM)

$$\Psi = \oint_S \mathbf{D} \cdot d\mathbf{S} = Q_{\text{enc}} = \int_V \rho_v \, dv$$

S is any closed surface, $d\mathbf{S}$ points outward and Q_{enc} is the total free charge inside (the last integral is written for a volume charge; enclosed point, line and surface charges count too). Always true, but it yields $\mathbf{D}$ only when symmetry makes $\mathbf{D}$ normal to the surface with constant magnitude (or tangential) on every part of S.

GAUSS'S LAW (POINT FORM)

$$\nabla \cdot \mathbf{D} = \rho_v, \qquad \nabla \cdot \mathbf{E} = \frac{\rho_v}{\varepsilon_0} \quad (\text{free space})$$

Follows from the integral form and the divergence theorem. To get the charge density from a given field, differentiate; do not integrate.

DIVERGENCE IN THE THREE COORDINATE SYSTEMS

$$\text{Cartesian:} \quad \nabla \cdot \mathbf{D} = \frac{\partial D_x}{\partial x} + \frac{\partial D_y}{\partial y} + \frac{\partial D_z}{\partial z}$$

$$\text{Cylindrical:} \quad \nabla \cdot \mathbf{D} = \frac{1}{\rho}\frac{\partial (\rho D_\rho)}{\partial \rho} + \frac{1}{\rho}\frac{\partial D_\phi}{\partial \phi} + \frac{\partial D_z}{\partial z}$$

$$\text{Spherical:} \quad \nabla \cdot \mathbf{D} = \frac{1}{r^2}\frac{\partial (r^2 D_r)}{\partial r} + \frac{1}{r \sin\theta}\frac{\partial (\sin\theta \, D_\theta)}{\partial \theta} + \frac{1}{r \sin\theta}\frac{\partial D_\phi}{\partial \phi}$$

The factors ρ, r^2 and $\sin\theta$ go INSIDE the derivative. A purely radial field with $D_r \propto 1/r^2$ has zero divergence everywhere except $r = 0$.

POTENTIAL DIFFERENCE

$$V_{AB} = V_A - V_B = -\int_B^A \mathbf{E} \cdot d\mathbf{l}$$

Work done per unit charge by an external agent in moving a positive test charge from B to A. Integrate along any convenient path; the lower limit is the reference point B.

WORK DONE IN MOVING A CHARGE

$$W_{\text{ext}} = -Q \int_A^B \mathbf{E} \cdot d\mathbf{l} = Q\,(V_B - V_A)$$

Work done by an external agent moving Q slowly (no change in kinetic energy) from A to B; the field itself does $-W_{\text{ext}}$. Positive when a positive charge is moved toward higher potential; zero along an equipotential and around any closed path.

POTENTIAL OF POINT CHARGES AND CONTINUOUS DISTRIBUTIONS

$$V = \frac{Q}{4\pi\varepsilon_0 r}, \qquad V(\mathbf{r}) = \sum_k \frac{Q_k}{4\pi\varepsilon_0 \left|\mathbf{r} - \mathbf{r}_k\right|}, \qquad V = \int \frac{dQ}{4\pi\varepsilon_0 R}$$

Reference at infinity. V is a scalar, so add the contributions with their signs and no components; this is often easier than integrating $\mathbf{E}$ directly, and $-\nabla V$ then gives the field.

FIELD FROM POTENTIAL (GRADIENT)

$$\mathbf{E} = -\nabla V$$

$$\text{Cartesian:} \quad \nabla V = \frac{\partial V}{\partial x}\hat{a}_x + \frac{\partial V}{\partial y}\hat{a}_y + \frac{\partial V}{\partial z}\hat{a}_z$$

$$\text{Cylindrical:} \quad \nabla V = \frac{\partial V}{\partial \rho}\hat{a}_\rho + \frac{1}{\rho}\frac{\partial V}{\partial \phi}\hat{a}_\phi + \frac{\partial V}{\partial z}\hat{a}_z$$

$$\text{Spherical:} \quad \nabla V = \frac{\partial V}{\partial r}\hat{a}_r + \frac{1}{r}\frac{\partial V}{\partial \theta}\hat{a}_\theta + \frac{1}{r \sin\theta}\frac{\partial V}{\partial \phi}\hat{a}_\phi$$

The minus sign makes $\mathbf{E}$ point toward decreasing V. $|\nabla V|$ is the largest rate of change of V with distance at that point.

POTENTIAL DIFFERENCE NEAR AN INFINITE LINE CHARGE

$$V_{AB} = V_A - V_B = \frac{\rho_L}{2\pi\varepsilon_0} \ln \frac{\rho_B}{\rho_A}$$

ρ_A and ρ_B are the perpendicular distances of A and B from the line. The reference cannot be at infinity (the logarithm diverges); only differences are meaningful.

POTENTIAL OF A UNIFORMLY CHARGED SPHERICAL SHELL AND SOLID SPHERE

$$\text{shell: } V = \begin{cases} \dfrac{Q}{4\pi\varepsilon_0 a}, & r \le a \\ \dfrac{Q}{4\pi\varepsilon_0 r}, & r \ge a \end{cases} \qquad \text{solid: } V = \begin{cases} \dfrac{Q\left(3a^2 - r^2\right)}{8\pi\varepsilon_0 a^3}, & r \le a \\ \dfrac{Q}{4\pi\varepsilon_0 r}, & r \ge a \end{cases}$$

Radius a, total charge Q, reference at infinity; the shell result also holds for an isolated charged conducting sphere. Inside the shell V is constant and equal to its surface value, although $\mathbf{E} = 0$ there. At the centre of the solid sphere $V = 3Q/(8\pi\varepsilon_0 a)$, 1.5 times its surface value. Stored energy: $Q^2/(8\pi\varepsilon_0 a)$ for the shell or conducting sphere, $3Q^2/(20\pi\varepsilon_0 a)$ for the uniformly charged solid sphere.

ELECTRIC DIPOLE

$$V = \frac{p\cos\theta}{4\pi\varepsilon_0 r^2}, \qquad \mathbf{E} = \frac{p}{4\pi\varepsilon_0 r^3}\left(2\cos\theta\,\hat{a}_r + \sin\theta\,\hat{a}_\theta\right)$$

Charges $+Q$ and $-Q$ a distance d apart, centred on the origin along the z-axis; the dipole moment $\mathbf{p} = Qd\,\hat{a}_z$ points from $-Q$ to $+Q$, and θ is measured from $+z$. Valid for $r \gg d$. V falls as $1/r^2$ and E as $1/r^3$; $V = 0$ on the whole plane $\theta = 90^\circ$. In a uniform external field $\mathbf{E}_0$ a dipole feels no net force, but a torque $\mathbf{T} = \mathbf{p} \times \mathbf{E}_0$, and has potential energy $U = -\mathbf{p} \cdot \mathbf{E}_0$, lowest when $\mathbf{p}$ is aligned with $\mathbf{E}_0$.

ELECTROSTATIC ENERGY

$$W_E = \frac{1}{2}\sum_{k=1}^{n} Q_k V_k \qquad \text{(point charges)}$$

$$W_E = \frac{1}{2}\int \rho_v V\, dv = \frac{1}{2}\int \mathbf{D}\cdot\mathbf{E}\, dv \qquad \text{(continuous charge, linear medium)}$$

$$w_E = \frac{1}{2}\mathbf{D}\cdot\mathbf{E} = \frac{1}{2}\varepsilon E^2 = \frac{D^2}{2\varepsilon} \qquad (\mathrm{J/m^3})$$

In the sum, V_k is the potential at Q_k due to all the OTHER charges; for two point charges $W_E = \dfrac{Q_1 Q_2}{4\pi\varepsilon_0 R}$. The first integral runs over the charge, the second over all space where the field exists. Applied to point charges, the integral forms would include each charge's infinite self-energy, which the sum leaves out. w_E is the energy density, to be integrated over the whole region where the field exists.

POISSON'S AND LAPLACE'S EQUATIONS

$$\nabla^2 V = -\frac{\rho_v}{\varepsilon} \quad \text{(Poisson)}, \qquad \nabla^2 V = 0 \quad \text{(Laplace, charge-free region)}$$

From $\nabla\cdot(\varepsilon\mathbf{E}) = \rho_v$ and $\mathbf{E} = -\nabla V$ in a homogeneous medium. Cartesian: $\nabla^2 V = \dfrac{\partial^2 V}{\partial x^2} + \dfrac{\partial^2 V}{\partial y^2} + \dfrac{\partial^2 V}{\partial z^2}$. When V depends only on the radius: $\dfrac{1}{\rho}\dfrac{d}{d\rho}\left(\rho\dfrac{dV}{d\rho}\right)$ (cylindrical) and $\dfrac{1}{r^2}\dfrac{d}{dr}\left(r^2\dfrac{dV}{dr}\right)$ (spherical). When V depends on one coordinate only, Laplace's equation integrates directly: $V = Ax + B$ (planar), $V = A\ln\rho + B$ (cylindrical), $V = A/r + B$ (spherical), with A and B fixed by the two boundary potentials. So between parallel plates with one homogeneous, charge-free dielectric in the gap (fringing neglected), V is linear in distance.

POLARISATION AND PERMITTIVITY

$$\mathbf{D} = \varepsilon_0\mathbf{E} + \mathbf{P}, \qquad \mathbf{P} = \varepsilon_0\chi_e\mathbf{E}, \qquad \varepsilon_r = 1 + \chi_e$$

χ_e is the electric susceptibility (zero in vacuum). The last two relations hold for linear isotropic media; $\varepsilon = \varepsilon_0\varepsilon_r$ is the permittivity.

BOUND (POLARISATION) CHARGE DENSITIES

$$\rho_{ps} = \mathbf{P} \cdot \hat{a}_n, \qquad \rho_{pv} = -\nabla \cdot \mathbf{P}$$

$\hat{a}_n$ is the outward normal of the dielectric's surface. The total bound charge of a dielectric body is zero. Gauss's law in terms of total charge: $\nabla \cdot (\varepsilon_0 \mathbf{E}) = \rho_v + \rho_{pv}$.

BOUNDARY CONDITIONS AT AN INTERFACE

$$E_{t1} = E_{t2}, \qquad D_{n1} - D_{n2} = \rho_S \quad (\hat{a}_n \cdot (\mathbf{D}_1 - \mathbf{D}_2) = \rho_S)$$

$\hat{a}_n$ is the unit normal pointing from region 2 into region 1, and ρ_S is the FREE surface charge on the interface. Between two perfect dielectrics $\rho_S = 0$, so D_n is continuous and $\varepsilon_1 E_{n1} = \varepsilon_2 E_{n2}$. Tangential $\mathbf{E}$ is always continuous; tangential $\mathbf{D}$ is not (unless $\varepsilon_1 = \varepsilon_2$): $D_{t1}/\varepsilon_1 = D_{t2}/\varepsilon_2$.

REFRACTION OF FIELD LINES

$$\frac{\tan\theta_1}{\tan\theta_2} = \frac{\varepsilon_{r1}}{\varepsilon_{r2}}$$

Angles measured from the NORMAL to the interface; valid when $\rho_S = 0$. The field line lies farther from the normal in the medium with the larger permittivity.

CAPACITANCE AND STORED ENERGY

$$C = \frac{Q}{V}, \qquad W_E = \frac{1}{2}CV^2 = \frac{Q^2}{2C} = \frac{1}{2}QV$$

Equal to $\frac{1}{2}\int \varepsilon E^2 \, dv$ over the region between the conductors. Capacitances in parallel add; in series their reciprocals add.

PARALLEL-PLATE CAPACITOR

$$C = \frac{\varepsilon A}{d}, \qquad E = \frac{V}{d}, \qquad \rho_S = \frac{\varepsilon V}{d}$$

Plate area A, gap d, one homogeneous dielectric; uniform field, fringing neglected.

PARALLEL PLATES, DIELECTRIC LAYERS STACKED ACROSS THE GAP

$$C = \frac{A}{\dfrac{d_1}{\varepsilon_1} + \dfrac{d_2}{\varepsilon_2}}, \qquad \frac{E_1}{E_2} = \frac{\varepsilon_2}{\varepsilon_1}$$

Layers parallel to the plates act as capacitors in series. D is the same in every layer, so the field is larger in the layer of lower permittivity. For any number of layers, $C = A/\sum_k (d_k/\varepsilon_k)$. Special cases in an air gap d: a dielectric slab of thickness t and relative permittivity ε_r, parallel to the plates, gives $C = \varepsilon_0 A/(d - t + t/\varepsilon_r)$; an uncharged metal slab of thickness t gives $C = \varepsilon_0 A/(d - t)$. Neither depends on where the slab sits in the gap.

PARALLEL PLATES, DIELECTRICS SIDE BY SIDE

$$C = \frac{\varepsilon_1 A_1 + \varepsilon_2 A_2}{d}$$

Each dielectric fills the whole gap d over part of the plate area ($A_1 + A_2 = A$): capacitors in parallel. $E = V/d$ is the same in both; D differs.

COAXIAL CAPACITOR

$$C = \frac{2\pi\varepsilon L}{\ln(b/a)}, \qquad E_\rho = \frac{V}{\rho \ln(b/a)}$$

Inner radius a, outer radius b, length L. The field is largest at the surface of the inner conductor, $\rho = a$.

SPHERICAL CAPACITOR AND ISOLATED SPHERE

$$C = \frac{4\pi\varepsilon}{\dfrac{1}{a} - \dfrac{1}{b}} = \frac{4\pi\varepsilon ab}{b-a}, \qquad C_{\text{isolated}} = 4\pi\varepsilon a$$

Concentric spheres of radii $a < b$; letting $b \to \infty$ gives the isolated sphere. With two layers (ε_1 for $a < r < c$, ε_2 for $c < r < b$): $C = 4\pi \Big/ \left[\frac{1}{\varepsilon_1}\left(\frac{1}{a} - \frac{1}{c}\right) + \frac{1}{\varepsilon_2}\left(\frac{1}{c} - \frac{1}{b}\right)\right]$.

TWO-WIRE LINE AND WIRE ABOVE GROUND (PER UNIT LENGTH)

$$C'_{\text{two-wire}} = \frac{\pi\varepsilon}{\cosh^{-1}(d/2a)} \approx \frac{\pi\varepsilon}{\ln(d/a)}, \qquad C'_{\text{wire-plane}} = \frac{2\pi\varepsilon}{\cosh^{-1}(h/a)} \approx \frac{2\pi\varepsilon}{\ln(2h/a)}$$

In F/m. Wire radius a, centre-to-centre spacing d, height h of the wire axis above a grounded plane; the approximations need $d \gg a$ and $h \gg a$. The wire-plane result follows from the image method: the plane lies midway between the wire and its image, at half the voltage, so the capacitance doubles.

INDUCED SURFACE CHARGE AND FORCE ON THE CHARGE

$$\rho_S = -\frac{Q\,h}{2\pi\left(x^2 + y^2 + h^2\right)^{3/2}}, \qquad \mathbf{F} = -\frac{Q^2}{4\pi\varepsilon_0(2h)^2}\,\hat{a}_z = -\frac{Q^2}{16\pi\varepsilon_0 h^2}\,\hat{a}_z$$

$\rho_S = \varepsilon_0 E_z$ at $z = 0^+$. Its magnitude is largest, $Q/(2\pi h^2)$, directly below the charge, and it integrates to $-Q$ over the whole plane. The force on Q is the Coulomb attraction of its image, a distance $2h$ away.

Magnetostatics, magnetic circuits and inductance

PERMEABILITY AND THE CONSTITUTIVE RELATION

$$\mathbf{B} = \mu\mathbf{H} = \mu_0\mu_r\mathbf{H}, \qquad \mathbf{B} = \mu_0(\mathbf{H} + \mathbf{M}), \qquad \mathbf{M} = \chi_m\mathbf{H}, \qquad \mu_r = 1 + \chi_m$$

Linear, isotropic medium. $\mathbf{M}$ is the magnetization (magnetic dipole moment per unit volume, A/m) and χ_m the magnetic susceptibility. In a ferromagnetic material μ is not a constant: it depends on H and on the magnetic history (see the B-H curve below).

MAGNETIC FLUX

$$\Phi = \int_S \mathbf{B}\cdot d\mathbf{S} \ \ (\text{Wb}), \qquad 1\ \text{T} = 1\ \text{Wb/m}^2 = 10^4\ \text{gauss}$$

$d\mathbf{S}$ is normal to the surface. For a uniform B normal to a flat area A, $\Phi = BA$.

BIOT-SAVART LAW

$$d\mathbf{H} = \frac{I\,d\mathbf{l}\times\hat{a}_R}{4\pi R^2} = \frac{I\,d\mathbf{l}\times\mathbf{R}}{4\pi R^3}, \qquad \mathbf{H} = \oint \frac{I\,d\mathbf{l}\times\hat{a}_R}{4\pi R^2}$$

Steady current. $\mathbf{R}$ points FROM the current element TO the field point. For a surface current density $\mathbf{K}$ (A/m) replace $I\,d\mathbf{l}$ by $\mathbf{K}\,dS$; for a volume current density $\mathbf{J}$ (A/m²) by $\mathbf{J}\,dv$. A point on the line of a straight segment (or its extension) gets no field from that segment, since $d\mathbf{l}\times\hat{a}_R = 0$.

FINITE STRAIGHT WIRE

$$\mathbf{H} = \frac{I}{4\pi\rho}\left(\sin\alpha_1 + \sin\alpha_2\right)\hat{a}_\phi$$

ρ is the perpendicular distance from the point to the line of the wire. α_1, α_2 are the angles, at the point, between that perpendicular and the lines to the two ends, when the ends lie on opposite sides of the foot of the perpendicular. If both ends are on the same side, use $\sin\alpha_2 - \sin\alpha_1$ with α_2 the larger angle. Direction by the right-hand rule.

INFINITE AND SEMI-INFINITE STRAIGHT WIRE

$$\mathbf{H} = \frac{I}{2\pi\rho}\hat{a}_\phi, \quad \mathbf{B} = \frac{\mu_0 I}{2\pi\rho}\hat{a}_\phi \ \text{(infinite)}; \qquad H = \frac{I}{4\pi\rho} \ \text{(semi-infinite, point level with its end)}$$

Infinite wire: $\alpha_1 = \alpha_2 = 90^\circ$. Semi-infinite wire with the point on the perpendicular through its finite end: $\alpha_1 = 0$, $\alpha_2 = 90^\circ$.

CIRCULAR LOOP, ON ITS AXIS

$$\mathbf{H} = \frac{Ia^2}{2\left(a^2+z^2\right)^{3/2}}\hat{a}_z, \qquad H_{\text{centre}} = \frac{I}{2a}, \qquad H_{\text{centre}} = \frac{NI}{2a} \ (N \text{ turns})$$

Loop of radius a in the xy-plane centred at the origin, current counter-clockwise seen from $+z$; z is the distance along the axis. Far away ($z \gg a$), $H \approx Ia^2/(2z^3)$, the on-axis field of a magnetic dipole.

ARC, RECTANGULAR AND SQUARE LOOP, AT THE CENTRE

$$H_{\text{arc}} = \frac{I\theta}{4\pi a} \ (\theta \text{ in rad}), \qquad H_{\text{semicircle}} = \frac{I}{4a}, \qquad H_{\text{rectangle}} = \frac{2I\sqrt{s_1^2+s_2^2}}{\pi s_1 s_2}, \qquad H_{\text{square}} = \frac{2\sqrt{2}\,I}{\pi s}$$

Arc of radius a subtending angle θ at its centre (radial leads to the centre add nothing). Rectangle of sides s_1 and s_2: four finite wires, all giving fields in the same direction at the centre; each side of length s_1 is at $\rho = s_2/2$ with $\sin\alpha_1 = \sin\alpha_2 = s_1/\sqrt{s_1^2+s_2^2}$, and likewise for the other pair. Square of side s ($s_1 = s_2 = s$): each side has $\rho = s/2$ and $\alpha_1 = \alpha_2 = 45^\circ$.

SOLENOID, ON ITS AXIS

$$H = \frac{nI}{2}\left(\frac{d_1}{\sqrt{d_1^2+a^2}} + \frac{d_2}{\sqrt{d_2^2+a^2}}\right), \qquad H_{\text{long, inside}} = nI, \qquad H_{\text{long, at an end}} = \frac{nI}{2}$$

$n = N/l$ turns per metre, radius a, point on the axis inside the solenoid at axial distances d_1 and d_2 from the two ends (for a point on the axis beyond one end, subtract the near-end term from the far-end term). A long, closely wound solenoid has a uniform $H = nI$ inside and $H \approx 0$ outside.

TOROID

$$H = \frac{NI}{2\pi\rho} \ (\text{inside the core}), \qquad H = 0 \ (\text{outside})$$

N closely wound turns; ρ is the distance from the toroid's axis. H does not depend on the core material; $B = \mu NI/(2\pi\rho)$ does.

INFINITE CURRENT SHEET

$$\mathbf{H} = \frac{1}{2}\mathbf{K} \times \hat{a}_n$$

Uniform surface current density $\mathbf{K}$ (A/m); $\hat{a}_n$ is the unit normal from the sheet towards the field point. $|\mathbf{H}| = K/2$ at any distance. Two parallel sheets carrying $+\mathbf{K}$ and $-\mathbf{K}$: $H = K$ between them and 0 outside.

AMPERE'S CIRCUITAL LAW (INTEGRAL FORM)

$$\oint_C \mathbf{H}\cdot d\mathbf{l} = I_{\text{enc}} = \int_S \mathbf{J}\cdot d\mathbf{S}$$

Steady currents. S is any surface bounded by the closed path C; the positive sense of I_{enc} follows the right-hand rule on the direction of travel round C. For N turns threading the path, $I_{\text{enc}} = NI$. Always true for steady currents, but it yields H directly only when symmetry lets a path be chosen along which H is tangential and constant in magnitude, and elsewhere perpendicular to the path or zero: infinite wire, round conductor, coaxial cable, current sheet, solenoid, toroid.

AMPERE'S LAW (POINT FORM) AND CURRENT CONTINUITY

$$\nabla \times \mathbf{H} = \mathbf{J}, \qquad \nabla\cdot\mathbf{J} = \nabla\cdot(\nabla\times\mathbf{H}) = 0$$

Magnetostatics only (time-varying fields add the displacement current density $\partial\mathbf{D}/\partial t$). In a current-free region $\nabla\times\mathbf{H} = 0$.

GAUSS'S LAW FOR MAGNETISM

$$\nabla \cdot \mathbf{B} = 0, \qquad \oint_S \mathbf{B} \cdot d\mathbf{S} = 0$$

No isolated magnetic poles: flux lines close on themselves and the net flux out of any closed surface is zero. Holds for time-varying fields too. It is the flux-junction rule of magnetic circuits.

CURL IN CARTESIAN COORDINATES

$$\nabla \times \mathbf{H} = \left(\frac{\partial H_z}{\partial y} - \frac{\partial H_y}{\partial z}\right)\hat{a}_x + \left(\frac{\partial H_x}{\partial z} - \frac{\partial H_z}{\partial x}\right)\hat{a}_y + \left(\frac{\partial H_y}{\partial x} - \frac{\partial H_x}{\partial y}\right)\hat{a}_z$$

The curl is circulation per unit area: $(\nabla \times \mathbf{H}) \cdot \hat{a}_n = \lim_{\Delta S \to 0} \oint \mathbf{H} \cdot d\mathbf{l} / \Delta S$, with the small path and $\hat{a}_n$ related by the right-hand rule.

STOKES' THEOREM

$$\oint_C \mathbf{H} \cdot d\mathbf{l} = \int_S (\nabla \times \mathbf{H}) \cdot d\mathbf{S}$$

Links the two forms of Ampere's law. S is any open surface bounded by C, with $d\mathbf{S}$ oriented by the right-hand rule on the direction of travel round C.

SOLID ROUND CONDUCTOR (UNIFORM CURRENT)

$$H_\phi = \frac{I\rho}{2\pi a^2} \ \ (\rho \le a), \qquad H_\phi = \frac{I}{2\pi\rho} \ \ (\rho \ge a)$$

Radius a, current I spread uniformly ($J = I/\pi a^2$, DC). H rises linearly from zero on the axis to its maximum $I/(2\pi a)$ at the surface, then falls as $1/\rho$. Inside a hollow tube whose current flows only in the wall, $H = 0$.

HYSTERESIS LOSS FROM THE LOOP

$$w_h = \oint H\, dB \ \ (\mathrm{J/m^3 \ per \ cycle}), \qquad P_h = w_h\, f\, V$$

The energy lost per cycle per unit volume equals the area enclosed by the B-H loop; multiply by the frequency f and the core volume V for the power.

EMPIRICAL CORE-LOSS RELATIONS

$$P_h = k_h\, f\, B_m^{\,x}\, V, \qquad P_e = k_e\, f^2 B_m^2 t^2\, V$$

B_m = peak flux density, t = lamination thickness, k_h and k_e material constants, x an empirical exponent that depends on the material (1.6 is the classical value). At a fixed B_m, hysteresis loss is proportional to f and eddy-current loss to f^2. Building the core from thin insulated laminations cuts eddy loss through the t^2 factor; it does not reduce hysteresis loss.

LORENTZ FORCE

$$\mathbf{F} = q\,(\mathbf{E} + \mathbf{u} \times \mathbf{B})$$

q carries its sign (an electron has $q = -e = -1.602 \times 10^{-19}$ C). The magnetic part $q\,\mathbf{u} \times \mathbf{B}$ is perpendicular to the velocity, so it does no work: it changes the direction of motion, never the speed or the kinetic energy.

CHARGED PARTICLE IN A UNIFORM B

$$r = \frac{m u_\perp}{|q|B}, \qquad \omega_c = \frac{|q|B}{m}, \qquad T = \frac{2\pi m}{|q|B}, \qquad \text{pitch} = u_\parallel T$$

$u_\perp$ and $u_\parallel$ are the velocity components perpendicular and parallel to $\mathbf{B}$; with $\mathbf{E} = 0$ the path is a circle (if $u_\parallel = 0$) or a helix. The period and angular frequency do not depend on the speed (non-relativistic motion). A charge accelerated from rest through a potential difference V has $u = \sqrt{2|q|V/m}$, so entering perpendicular to $\mathbf{B}$ it moves on a circle of radius $r = \sqrt{2mV/|q|}\,/B$. In crossed $\mathbf{E}$ and $\mathbf{B}$ (both perpendicular to $\mathbf{u}$ and to each other, forces opposed), a charge passes undeflected when $u = E/B$.

FORCE ON A CURRENT ELEMENT AND ON A STRAIGHT CONDUCTOR

$$d\mathbf{F} = I\,d\mathbf{l} \times \mathbf{B}, \qquad \mathbf{F} = I\,\mathbf{L} \times \mathbf{B}, \qquad |\mathbf{F}| = BIL\sin\theta$$

$\mathbf{L}$ points along the current and θ is the angle between the conductor and $\mathbf{B}$; the second and third forms need a straight conductor in a uniform $\mathbf{B}$. A closed loop of any shape in a UNIFORM field feels zero net force (it can still feel a torque).

HALL EFFECT

$$E_H = u_d B, \qquad V_H = \frac{BI}{n|q|t}, \qquad R_H = \frac{1}{nq}$$

A slab carries a current I perpendicular to a uniform $\mathbf{B}$; t is the slab's thickness measured along $\mathbf{B}$, n the number density of the charge carriers and q their charge (one type of carrier). The magnetic force pushes the carriers, drifting at speed u_d, towards one side until the transverse Hall field E_H balances it. V_H does not depend on the slab's width. The Hall coefficient R_H (m³/C) is negative for electrons and positive for holes, so the polarity of V_H identifies the carrier type; because $V_H \propto 1/n$, it is far larger in a semiconductor than in a metal.

FORCE BETWEEN TWO LONG PARALLEL WIRES

$$\frac{F}{l} = \frac{\mu_0 I_1 I_2}{2\pi d}\ \ \mathrm{N/m}$$

Infinitely long parallel wires with separation d, in air. Currents in the same direction attract; in opposite directions they repel. It follows from $\mathbf{F}_2 = I_2\mathbf{L} \times \mathbf{B}_1$ with $B_1 = \mu_0 I_1/(2\pi d)$.

MAGNETIC DIPOLE MOMENT, TORQUE AND ENERGY

$$\mathbf{m} = NIS\,\hat{a}_n, \qquad \mathbf{T} = \mathbf{m} \times \mathbf{B}, \qquad |\mathbf{T}| = NISB\sin\theta, \qquad U = -\mathbf{m}\cdot\mathbf{B}$$

Planar loop of area S with N turns carrying I; $\hat{a}_n$ is the normal given by the right-hand rule on the current; $\mathbf{m}$ in A·m². Uniform $\mathbf{B}$; θ is the angle between $\hat{a}_n$ and $\mathbf{B}$, so the torque is largest when the PLANE of the loop is parallel to $\mathbf{B}$. Stable equilibrium (minimum U, zero torque) is $\mathbf{m}$ parallel to $\mathbf{B}$.

FIELD OF A MAGNETIC DIPOLE (FAR FIELD)

$$\mathbf{A} = \frac{\mu_0\,\mathbf{m} \times \hat{a}_r}{4\pi r^2}, \qquad \mathbf{B} = \frac{\mu_0 m}{4\pi r^3}\left(2\cos\theta\,\hat{a}_r + \sin\theta\,\hat{a}_\theta\right)$$

Small current loop at the origin with $\mathbf{m} = m\,\hat{a}_z$, spherical coordinates, r much larger than the loop. On the axis ($\theta = 0$), $B = \mu_0 m/(2\pi r^3)$, which matches the loop-axis formula for $z \gg a$.

FIELD OF AN ELECTRIC DIPOLE (FAR FIELD)

$$V = \frac{p\cos\theta}{4\pi\varepsilon_0 r^2}, \qquad \mathbf{E} = \frac{p}{4\pi\varepsilon_0 r^3}\left(2\cos\theta\,\hat{a}_r + \sin\theta\,\hat{a}_\theta\right)$$

Charges $+Q$ and $-Q$ separated by d, with $\mathbf{p} = Q\mathbf{d}$ directed from $-Q$ to $+Q$ along $\hat{a}_z$; $r \gg d$; $\varepsilon_0 = 8.854 \times 10^{-12}$ F/m. In a uniform $\mathbf{E}$: torque $\mathbf{T} = \mathbf{p} \times \mathbf{E}$, energy $U = -\mathbf{p}\cdot\mathbf{E}$.

NORMAL COMPONENT OF B

$$B_{1n} = B_{2n} \quad\Longrightarrow\quad \mu_1 H_{1n} = \mu_2 H_{2n}$$

Always, from $\nabla\cdot\mathbf{B} = 0$. The normal component of $\mathbf{H}$ is therefore discontinuous whenever $\mu_1 \neq \mu_2$.

TANGENTIAL COMPONENT OF H

$$\hat{a}_{n12} \times (\mathbf{H}_1 - \mathbf{H}_2) = \mathbf{K}; \qquad \mathbf{K} = 0:\ \ H_{1t} = H_{2t},\ \ \frac{B_{1t}}{\mu_1} = \frac{B_{2t}}{\mu_2}$$

$\hat{a}_{n12}$ points from medium 2 into medium 1. The tangential $\mathbf{H}$ jumps by $|\mathbf{K}|$ in the tangential component perpendicular to $\mathbf{K}$. With no surface current the tangential $\mathbf{H}$ is continuous and the tangential $\mathbf{B}$ changes in the ratio of the permeabilities.

REFRACTION OF B LINES

$$\frac{\tan\theta_1}{\tan\theta_2} = \frac{\mu_1}{\mu_2}$$

Linear isotropic media, no surface current; θ_1, θ_2 are measured from the NORMAL. Going from air into iron ($\mu_2 \gg \mu_1$) the lines bend towards the surface inside the iron; coming out of iron into air they leave almost normal to the iron surface, unless they run exactly along it inside.

MMF, RELUCTANCE AND FLUX

$$\mathcal{F} = NI \ \text{(A-t)}, \qquad \mathcal{R} = \frac{l}{\mu_0\mu_r A} \ (\text{A-t/Wb} = \text{H}^{-1}), \qquad \Phi = \frac{\mathcal{F}}{\mathcal{R}}, \qquad \mathcal{P} = \frac{1}{\mathcal{R}}$$

l = mean length of the flux path in that part, A = its cross-sectional area, $\mathcal{P}$ = permeance (H). Valid while the material is linear, B is uniform over the cross-section and leakage flux is neglected.

LOOP AND JUNCTION LAWS FOR MAGNETIC CIRCUITS

$$\sum \mathcal{F} = \sum_k H_k l_k = \sum_k \Phi_k \mathcal{R}_k \ \text{(any closed path)}, \qquad \sum \Phi = 0 \ \text{(any junction)}$$

The loop law is Ampere's circuital law; the junction law is $\oint \mathbf{B}\cdot d\mathbf{S} = 0$. Parts in series carry one flux and $\mathcal{R} = \mathcal{R}_1 + \mathcal{R}_2 + \ldots$. Parallel limbs between the same two junctions have equal mmf drops; their combined reluctance follows $1/\mathcal{R} = 1/\mathcal{R}_1 + 1/\mathcal{R}_2$, and the flux divides between them in inverse ratio of their reluctances.

AIR GAP AND FRINGING

$$\mathcal{R}_g = \frac{l_g}{\mu_0 A_g}, \qquad A_g \approx (a + l_g)(b + l_g) \ \text{(fringing correction for an } a \times b \text{ face)}$$

Air gap of length l_g. With fringing neglected, A_g equals the core area. Fringing spreads the flux over a larger area, lowering the gap's reluctance and its flux density; the $(a + l_g)(b + l_g)$ rule is an approximate correction for a short gap between rectangular faces. The gap's reluctance exceeds that of an iron path of length l_c and the same area whenever $l_g > l_c/\mu_r$, which for a core with a large μ_r is a very short gap.

INDUCTANCE FROM RELUCTANCE

$$L = \frac{N\Phi}{I} = \frac{N^2}{\mathcal{R}_{\text{total}}} = N^2\mathcal{P}, \qquad M = \frac{N_1 N_2}{\mathcal{R}} \ \text{(two coils sharing one flux path)}$$

Linear core, every turn linked by the same flux. $L \propto N^2$. When the gap dominates, $\mathcal{R}_{\text{total}} \approx \mathcal{R}_g$ and $L \approx \mu_0 N^2 A_g / l_g$, nearly independent of the core's (variable) permeability, which is one reason inductor cores are gapped. Two coils of N_1 and N_2 turns on the same core with no leakage: $L_1 = N_1^2/\mathcal{R}$, $L_2 = N_2^2/\mathcal{R}$ and $M = N_1 N_2/\mathcal{R}$, so $M = \sqrt{L_1 L_2}$ and $k = 1$.

ENERGY IN A MAGNETIC CIRCUIT AND PULL OF AN ELECTROMAGNET

$$W = \frac{1}{2}\mathcal{F}\Phi = \frac{1}{2}\Phi^2\mathcal{R} = \frac{1}{2}LI^2, \qquad F = \frac{B^2 A}{2\mu_0} \ \text{per air-gap face}$$

Linear circuit. At the same B, the energy density $B^2/(2\mu_0)$ in air is μ_r times that in iron, so the gap holds much of the energy. The pull on a pole face of area A across a gap with uniform B (no fringing) follows from virtual work at constant flux; a U-shaped magnet lifting a bar across two gaps exerts twice this.

SELF INDUCTANCE

$$L = \frac{\lambda}{I} = \frac{N\Phi}{I} \ \text{(H)}, \qquad L = \frac{2W_m}{I^2}$$

Linear medium. For a nonlinear core the incremental inductance $d\lambda/di$ depends on the operating point.

MUTUAL INDUCTANCE

$$M_{21} = \frac{\lambda_{21}}{I_1} = \frac{N_2 \Phi_{21}}{I_1}, \qquad M_{12} = M_{21} = M, \qquad M = \frac{\mu_0}{4\pi} \oint_{C_1} \oint_{C_2} \frac{d\mathbf{l}_1 \cdot d\mathbf{l}_2}{R}$$

Φ_{21} is the flux produced by coil 1 that links each turn of coil 2. Reciprocity ($M_{12} = M_{21}$) holds in any linear medium, so compute M from whichever coil's field is easier to find. The double line integral is the general form for two thin single-turn loops in a homogeneous non-magnetic medium, R being the distance between the two elements.

COUPLING COEFFICIENT AND SERIES CONNECTION

$$k = \frac{M}{\sqrt{L_1 L_2}} \;\; (0 \le k \le 1), \qquad L_{\text{series}} = L_1 + L_2 \pm 2M, \qquad v_2 = M \frac{di_1}{dt}$$

Plus sign when the two fluxes aid (the current enters both dotted ends), minus when they oppose; hence $M = (L_{\text{aid}} - L_{\text{opp}})/4$. $k = 1$ only if every flux line of each coil links every turn of the other. The polarity of the induced voltage v_2 follows the dot convention.

STORED MAGNETIC ENERGY

$$W_m = \frac{1}{2} L I^2 = \frac{1}{2} \lambda I = \frac{1}{2} \int_V \mathbf{B} \cdot \mathbf{H} \, dv, \qquad w_m = \frac{1}{2} \mu H^2 = \frac{B^2}{2\mu} \;\; (\mathrm{J/m^3})$$

Linear medium. For two coupled coils, $W_m = \frac{1}{2} L_1 I_1^2 + \frac{1}{2} L_2 I_2^2 \pm M I_1 I_2$, plus when the fluxes aid.

LONG SOLENOID

$$L = \frac{\mu N^2 A}{l} = \mu n^2 A l$$

Length l much greater than the radius, uniform core of permeability μ filling it, A = cross-sectional area, $n = N/l$. End effects make the L of a real, finite solenoid somewhat smaller.

TOROID

$$L = \frac{\mu N^2 h}{2\pi} \ln \frac{b}{a} \;\; (\text{rectangular section}), \qquad L \approx \frac{\mu N^2 A}{2\pi R} \;\; (\text{thin toroid})$$

Rectangular cross-section of height h, inner radius a and outer radius b, from integrating $B = \mu N I/(2\pi\rho)$ across the section. Thin toroid: mean radius R much larger than the section's dimensions, A = section area; for a circular section of radius r, $L \approx \mu N^2 r^2/(2R)$.

COAXIAL LINE (PER UNIT LENGTH)

$$L'_{\text{ext}} = \frac{\mu}{2\pi} \ln \frac{b}{a}, \qquad L'_{\text{int}} = \frac{\mu_c}{8\pi}, \qquad L' = \frac{\mu_0}{8\pi} + \frac{\mu_0}{2\pi} \ln \frac{b}{a} \;\; (\mathrm{H/m})$$

Inner conductor radius a, outer conductor inner radius b. External part: flux in the dielectric of permeability μ. Internal part: solid inner conductor of permeability μ_c carrying a uniform (DC) current, found by the energy route. The last form is for non-magnetic conductors and dielectric with a thin outer conductor. At high frequency the current crowds to the surface (skin effect) and $L'_{\text{int}} \to 0$.

TWO-WIRE LINE (PER UNIT LENGTH)

$$L'_{\text{ext}} = \frac{\mu}{\pi} \cosh^{-1} \frac{d}{2a} \approx \frac{\mu}{\pi} \ln \frac{d}{a} \;\; (d \gg a), \qquad L' = \frac{\mu_0}{\pi} \left(\frac{1}{4} + \ln \frac{d}{a} \right) = \frac{\mu_0}{\pi} \ln \frac{d}{0.7788\, a}$$

Two parallel wires of radius a with centre-to-centre spacing d, carrying equal and opposite currents. The second form ($d \gg a$) adds the internal inductance $\mu_0/(8\pi)$ of EACH wire (non-magnetic, uniform current); $0.7788\, a = a\, e^{-1/4}$ is the geometric mean radius of a solid round wire.

MUTUAL INDUCTANCE: COIL ON A LONG SOLENOID; WIRE AND COPLANAR LOOP

$$M = \mu_0 n_1 N_2 A_{\text{inner}}, \qquad M_{\text{wire-loop}} = \frac{\mu_0 h}{2\pi} \ln \frac{d+w}{d}$$

First: a coil of N_2 turns coaxial with a long air-cored solenoid of n_1 turns/m and placed well inside its length, away from the ends; A_{inner} is the cross-section of whichever winding is inside, because the solenoid's field is confined to its own cross-section. Second: an infinitely long straight wire and a coplanar rectangular loop with sides h parallel and w perpendicular to the wire, the nearer side at distance d, in air.

Maxwell's equations, Faraday's law, boundary conditions and the Poynting vector

FARADAY'S LAW (FLUX RULE)

$$V_{\text{emf}} = -\frac{d\Psi}{dt} = -N\frac{d\Phi}{dt}, \qquad \Phi = \int_S \mathbf{B}\cdot d\mathbf{S}$$

$\Psi = N\Phi$ is the flux linkage of N tightly wound turns. Holds for any closed circuit of thin wire, stationary or moving, including a bar sliding on rails. Where the current path through a moving conductor is not a definite wire (a rotating disc), compute the emf from the $\mathbf{u}\times\mathbf{B}$ integral below instead. A positive emf drives current in the sense that is right-handed about the $d\mathbf{S}$ used to compute Φ.

TRANSFORMER EMF (STATIONARY PATH)

$$V_{\text{emf}}^{\text{tr}} = \oint_C \mathbf{E}\cdot d\mathbf{l} = -\int_S \frac{\partial \mathbf{B}}{\partial t}\cdot d\mathbf{S}$$

Path fixed in space, $\mathbf{B}$ changing with time.

MOTIONAL EMF

$$V_{\text{emf}}^{\text{m}} = \oint_C (\mathbf{u}\times\mathbf{B})\cdot d\mathbf{l}, \qquad V = BLu \ (\text{rod}, \mathbf{u}, \mathbf{B} \text{ mutually perpendicular})$$

Conductor moving with velocity $\mathbf{u}$ through $\mathbf{B}$; only the moving segments contribute. Between the ends a and b of an open-circuited rod, $V_b - V_a = \int_a^b (\mathbf{u}\times\mathbf{B})\cdot d\mathbf{l}$: the end towards which $\mathbf{u}\times\mathbf{B}$ points is positive.

FARADAY'S LAW, POINT FORM

$$\nabla\times\mathbf{E} = -\frac{\partial \mathbf{B}}{\partial t}$$

Stokes' theorem applied to the transformer emf; holds at every point. In static fields $\nabla\times\mathbf{E} = 0$ (E is conservative); with a time-varying $\mathbf{B}$ it is not.

COIL ROTATING IN A UNIFORM STATIC FIELD

$$\Phi = BA\cos\omega t \ \Rightarrow\ e = NBA\omega\sin\omega t, \qquad e_{\max} = NBA\omega$$

N-turn coil of area A rotating at ω rad/s about an axis perpendicular to $\mathbf{B}$, with the coil plane perpendicular to $\mathbf{B}$ at $t = 0$.

EMF OF A COIL LINKED BY A SINUSOIDAL FLUX

$$\Phi = \Phi_m \sin\omega t \ \Rightarrow\ e = -N\frac{d\Phi}{dt} = -N\omega\Phi_m\cos\omega t, \qquad E_{\text{rms}} = \frac{2\pi f N\Phi_m}{\sqrt{2}} \approx 4.44\, f N\Phi_m$$

Stationary N-turn coil, every turn linked by the same flux of peak value Φ_m at frequency f ($\omega = 2\pi f$). The peak emf is $2\pi f N\Phi_m$, and the emf lags the flux by 90°.

SELF- AND MUTUALLY INDUCED EMF

$$e = -\frac{d\Psi}{dt} = -L\frac{di}{dt}, \quad L = \frac{N\Phi}{i}; \qquad e_2 = -M\frac{di_1}{dt}, \quad M = \frac{N_2\Phi_{21}}{i_1}$$

Coils fixed in place in a linear magnetic medium. Φ is the flux that a coil's own current i sets up through each of its N turns; Φ_{21} is the flux produced by i_1 through each of the N_2 turns of coil 2, and e_2 is the emf induced in coil 2 by a changing i_1. The minus sign is Lenz's law. Treated as a circuit element with the passive sign convention, the voltage across an inductor is $v = L\,di/dt$.

ROD ROTATING ABOUT ONE END (FARADAY DISC)

$$V = \int_0^L \omega r B\, dr = \frac{1}{2}B\omega L^2$$

Rod of length L rotating at ω in a plane perpendicular to a uniform static $\mathbf{B}$, pivoted at one end; also the emf between axle and rim of a disc of radius L.

CONTINUITY EQUATION (CHARGE CONSERVATION)

$$\nabla \cdot \mathbf{J} = -\frac{\partial \rho_v}{\partial t}, \qquad \oint_S \mathbf{J}\cdot d\mathbf{S} = -\frac{dQ_{\text{enc}}}{dt}$$

Always true. For steady currents $\nabla \cdot \mathbf{J} = 0$, the field form of KCL.

DISPLACEMENT CURRENT

$$\mathbf{J}_d = \frac{\partial \mathbf{D}}{\partial t}, \qquad I_d = \int_S \frac{\partial \mathbf{D}}{\partial t}\cdot d\mathbf{S}$$

A/m²; exists wherever $\mathbf{D}$ changes with time, including in vacuum. Phasor form in a linear medium: $\mathbf{J}_{ds} = j\omega\varepsilon\mathbf{E}_s$.

AMPÈRE–MAXWELL LAW, POINT FORM

$$\nabla \times \mathbf{H} = \mathbf{J} + \frac{\partial \mathbf{D}}{\partial t}$$

$\mathbf{J}$ is the free (conduction or convection) current density; in a conductor $\mathbf{J} = \sigma\mathbf{E}$.

DISPLACEMENT CURRENT IN A PARALLEL-PLATE CAPACITOR

$$I_d = \varepsilon A\frac{d}{dt}\left(\frac{v}{d}\right) = C\frac{dv}{dt}, \qquad C = \frac{\varepsilon A}{d}$$

Plate area A, gap d, uniform field $E = v/d$ (fringing neglected); equals the conduction current in the leads.

CONDUCTION TO DISPLACEMENT CURRENT RATIO (LOSS TANGENT)

$$\frac{|\mathbf{J}_c|}{|\mathbf{J}_d|} = \frac{\sigma}{\omega\varepsilon}$$

Time-harmonic field in a medium with conductivity σ and real permittivity ε: $\mathbf{J}_c = \sigma\mathbf{E}_s$ and $\mathbf{J}_d = j\omega\varepsilon\mathbf{E}_s$ are 90° apart, with $\mathbf{J}_d$ leading.

MAXWELL'S EQUATIONS, POINT FORM

$$\nabla \cdot \mathbf{D} = \rho_v$$
$$\nabla \cdot \mathbf{B} = 0$$
$$\nabla \times \mathbf{E} = -\frac{\partial \mathbf{B}}{\partial t}$$
$$\nabla \times \mathbf{H} = \mathbf{J} + \frac{\partial \mathbf{D}}{\partial t}$$

General time-varying fields; ρ_v and $\mathbf{J}$ are free charge and free current densities.

MAXWELL'S EQUATIONS, INTEGRAL FORM

$$\oint_S \mathbf{D}\cdot d\mathbf{S} = Q_{\text{enc}} = \int_v \rho_v\, dv$$
$$\oint_S \mathbf{B}\cdot d\mathbf{S} = 0$$
$$\oint_C \mathbf{E}\cdot d\mathbf{l} = -\frac{d}{dt}\int_S \mathbf{B}\cdot d\mathbf{S}$$
$$\oint_C \mathbf{H}\cdot d\mathbf{l} = I_{\text{enc}} + \frac{d}{dt}\int_S \mathbf{D}\cdot d\mathbf{S}$$

Stationary surfaces and paths. In the last two lines S is any surface bounded by C, with $d\mathbf{S}$ right-handed about $d\mathbf{l}$, and $I_{\text{enc}} = \int_S \mathbf{J}\cdot d\mathbf{S}$.

CONSTITUTIVE RELATIONS

$$\mathbf{D} = \varepsilon\mathbf{E} = \varepsilon_0\varepsilon_r\mathbf{E}, \qquad \mathbf{B} = \mu\mathbf{H} = \mu_0\mu_r\mathbf{H}, \qquad \mathbf{J} = \sigma\mathbf{E}$$

Linear, isotropic, homogeneous media. In general $\mathbf{D} = \varepsilon_0\mathbf{E} + \mathbf{P}$ and $\mathbf{B} = \mu_0(\mathbf{H} + \mathbf{M})$.

LORENTZ FORCE

$$\mathbf{F} = q\,(\mathbf{E} + \mathbf{u}\times\mathbf{B})$$

Force on a point charge q moving with velocity $\mathbf{u}$; the magnetic part does no work because it is perpendicular to $\mathbf{u}$.

MAXWELL'S EQUATIONS, PHASOR (TIME-HARMONIC) FORM

$$\nabla\cdot\mathbf{D}_s = \rho_{vs}$$
$$\nabla\cdot\mathbf{B}_s = 0$$
$$\nabla\times\mathbf{E}_s = -j\omega\mu\mathbf{H}_s$$
$$\nabla\times\mathbf{H}_s = (\sigma + j\omega\varepsilon)\mathbf{E}_s$$

Fields varying as $e^{j\omega t}$ in a linear medium. The last line assumes the only current is conduction current $\sigma\mathbf{E}_s$ (no impressed source current).

GENERAL BOUNDARY CONDITIONS (VECTOR FORM)

$$\hat{a}_n\times(\mathbf{E}_1 - \mathbf{E}_2) = 0$$
$$\hat{a}_n\times(\mathbf{H}_1 - \mathbf{H}_2) = \mathbf{K}$$
$$\hat{a}_n\cdot(\mathbf{D}_1 - \mathbf{D}_2) = \rho_s$$
$$\hat{a}_n\cdot(\mathbf{B}_1 - \mathbf{B}_2) = 0$$

$\hat{a}_n$ is the unit normal pointing from medium 2 into medium 1. Valid for static and time-varying fields.

GENERAL BOUNDARY CONDITIONS (COMPONENT FORM)

$$E_{1t} = E_{2t}, \qquad B_{1n} = B_{2n}, \qquad D_{1n} - D_{2n} = \rho_s, \qquad \mathbf{H}_{1t} - \mathbf{H}_{2t} = \mathbf{K}\times\hat{a}_n$$

Normal components are taken along $\hat{a}_n$ (from 2 into 1). Only the tangential H component perpendicular to $\mathbf{K}$ jumps, and it jumps by the magnitude of $\mathbf{K}$.

DIELECTRIC–DIELECTRIC INTERFACE (NO FREE SURFACE CHARGE)

$$E_{1t} = E_{2t}, \qquad \varepsilon_1 E_{1n} = \varepsilon_2 E_{2n}, \qquad \frac{D_{1t}}{\varepsilon_1} = \frac{D_{2t}}{\varepsilon_2}$$

Two perfect dielectrics with $\rho_s = 0$: tangential E and normal D are continuous, while normal E and tangential D jump.

REFRACTION OF FIELD LINES AT A DIELECTRIC INTERFACE

$$\frac{\tan\theta_1}{\tan\theta_2} = \frac{\varepsilon_1}{\varepsilon_2} = \frac{\varepsilon_{r1}}{\varepsilon_{r2}}$$

θ measured from the normal, charge-free interface. The field makes the larger angle with the normal in the medium of higher permittivity.

MAGNETIC INTERFACE (NO SURFACE CURRENT)

$$H_{1t} = H_{2t}, \qquad \mu_1 H_{1n} = \mu_2 H_{2n}, \qquad \frac{\tan\theta_1}{\tan\theta_2} = \frac{\mu_1}{\mu_2}$$

Any interface between media of finite conductivity, where $\mathbf{K} = 0$; θ from the normal.

SURFACE OF A PERFECT ELECTRIC CONDUCTOR

$$\mathbf{E}_{1t} = 0, \qquad B_{1n} = 0, \qquad D_{1n} = \rho_s, \qquad \hat{a}_n \times \mathbf{H}_1 = \mathbf{K}$$

Medium 2 is a perfect conductor ($\sigma \to \infty$), inside which time-varying $\mathbf{E}, \mathbf{D}, \mathbf{B}, \mathbf{H}$ are all zero; $\hat{a}_n$ points out of the conductor. At the surface E is purely normal and H purely tangential.

WAVE EQUATION (TIME DOMAIN)

$$\nabla^2 \mathbf{E} = \mu\sigma \frac{\partial \mathbf{E}}{\partial t} + \mu\varepsilon \frac{\partial^2 \mathbf{E}}{\partial t^2}$$

Source-free, linear, homogeneous, isotropic medium; $\mathbf{H}$ obeys the same equation. Lossless ($\sigma = 0$): $\nabla^2 \mathbf{E} = \mu\varepsilon\, \partial^2 \mathbf{E}/\partial t^2$.

HELMHOLTZ (PHASOR WAVE) EQUATION

$$\nabla^2 \mathbf{E}_s - \gamma^2 \mathbf{E}_s = 0, \qquad \gamma^2 = j\omega\mu(\sigma + j\omega\varepsilon)$$

Time-harmonic fields ($e^{j\omega t}$) in the same medium; the same equation holds for $\mathbf{H}_s$.

PROPAGATION CONSTANT AND INTRINSIC IMPEDANCE

$$\gamma = \alpha + j\beta = \sqrt{j\omega\mu(\sigma + j\omega\varepsilon)}, \qquad \eta = \sqrt{\frac{j\omega\mu}{\sigma + j\omega\varepsilon}} = |\eta| \angle \theta_\eta$$

α (Np/m) is the attenuation constant and β (rad/m) the phase constant; take the root with $\alpha, \beta \geq 0$, and the root of η with a non-negative real part. $\tan 2\theta_\eta = \sigma/(\omega\varepsilon)$, so $\theta_\eta = 0$ in a lossless medium and $0^\circ < \theta_\eta < 45^\circ$ in a lossy one (approaching 45° in a good conductor); H lags E in time by θ_η.

LOSSLESS MEDIUM

$$\beta = \omega\sqrt{\mu\varepsilon}, \quad v_p = \frac{\omega}{\beta} = \frac{1}{\sqrt{\mu\varepsilon}} = \frac{c}{\sqrt{\mu_r \varepsilon_r}}, \quad \lambda = \frac{2\pi}{\beta}, \quad \eta = \sqrt{\frac{\mu}{\varepsilon}} = \eta_0 \sqrt{\frac{\mu_r}{\varepsilon_r}}$$

$\sigma = 0$, so $\alpha = 0$ and η is real: E and H are in phase.

UNIFORM PLANE WAVE TRAVELLING IN +Z

$$\mathbf{E}_s = E_0 e^{-\gamma z} \hat{a}_x \;\Rightarrow\; \mathbf{E}(z,t) = E_0 e^{-\alpha z} \cos(\omega t - \beta z)\, \hat{a}_x$$

E_0 real. A wave travelling in $-z$ has $e^{+\gamma z}$ and $\cos(\omega t + \beta z)$; its amplitude $E_0 e^{+\alpha z}$ decreases as it travels towards $-z$.

E AND H OF A UNIFORM PLANE WAVE

$$\mathbf{H}_s = \frac{1}{\eta}\, \hat{a}_k \times \mathbf{E}_s, \qquad \mathbf{E}_s = -\eta\, \hat{a}_k \times \mathbf{H}_s$$

Wave travelling along the unit vector $\hat{a}_k$ in a homogeneous medium: $\mathbf{E}$, $\mathbf{H}$ and $\hat{a}_k$ are mutually perpendicular, and $\mathbf{E} \times \mathbf{H}$ points along $\hat{a}_k$. Example: $\mathbf{E}$ along $\hat{a}_x$, travel along $\hat{a}_z$ gives $\mathbf{H}$ along $\hat{a}_y$.

PHASOR AND INSTANTANEOUS FIELD

$$\mathbf{E}(\mathbf{r}, t) = \mathrm{Re}\left[\mathbf{E}_s(\mathbf{r})\, e^{j\omega t}\right], \qquad \frac{\partial}{\partial t} \leftrightarrow j\omega, \qquad \frac{\partial^2}{\partial t^2} \leftrightarrow -\omega^2$$

Sinusoidal steady state at a single frequency; e.g. $\mathbf{E}_s = E_0 e^{-j\beta z} e^{j\phi} \hat{a}_x$ (E_0 real) corresponds to $E_0 \cos(\omega t - \beta z + \phi)\hat{a}_x$.

POYNTING VECTOR (INSTANTANEOUS)

$$\mathbf{S}(\mathbf{r}, t) = \mathbf{E}(\mathbf{r}, t) \times \mathbf{H}(\mathbf{r}, t) \quad (\mathrm{W/m^2})$$

Uses the real time-domain fields, never the phasors.

POYNTING'S THEOREM (INTEGRAL FORM)

$$\oint_S (\mathbf{E}\times\mathbf{H})\cdot d\mathbf{S} = -\frac{d}{dt}\int_v \left(\frac{1}{2}\varepsilon E^2 + \frac{1}{2}\mu H^2\right) dv - \int_v \sigma E^2\, dv$$

Fixed volume v bounded by the closed surface S (outward $d\mathbf{S}$), linear non-dispersive medium, and the only current inside is conduction current $\sigma\mathbf{E}$. The left side is the total power leaving through S; so net power flowing in = rate of rise of stored energy + ohmic loss. With an impressed source current $\mathbf{J}_s$ inside, the power it supplies, $-\int_v \mathbf{J}_s\cdot\mathbf{E}\,dv$, is added to the right side.

ENERGY DENSITIES

$$w_e = \frac{1}{2}\varepsilon E^2, \qquad w_m = \frac{1}{2}\mu H^2 \quad (\mathrm{J/m^3}); \qquad \langle w_e\rangle = \frac{1}{4}\varepsilon|\mathbf{E}_s|^2, \quad \langle w_m\rangle = \frac{1}{4}\mu|\mathbf{H}_s|^2$$

Linear medium: instantaneous values, then time averages for time-harmonic fields with peak phasors. For a single uniform plane wave in a lossless medium $w_e = w_m$ at every instant, because $H = E/\eta$.

TIME-AVERAGE POYNTING VECTOR

$$\mathbf{S}_{\text{avg}} = \frac{1}{T}\int_0^T \mathbf{S}\,dt = \frac{1}{2}\mathrm{Re}\left[\mathbf{E}_s\times\mathbf{H}_s^*\right]$$

Time-harmonic fields with peak-value phasors; with rms phasors drop the $\frac{1}{2}$. Since only the real part is kept, $\mathrm{Re}[\mathbf{E}_s\times\mathbf{H}_s^*] = \mathrm{Re}[\mathbf{E}_s^*\times\mathbf{H}_s]$, but one of the two must be conjugated.

AVERAGE POWER DENSITY OF A PLANE WAVE, LOSSLESS MEDIUM

$$\mathbf{S}_{\text{avg}} = \frac{E_0^2}{2\eta}\,\hat{a}_k = \frac{1}{2}\eta H_0^2\,\hat{a}_k$$

E_0 and $H_0 = E_0/\eta$ are peak amplitudes; in free space $\eta = \eta_0 \approx 120\pi\ \Omega$. For a wave along z of any polarisation, replace E_0^2 by $|\mathbf{E}_s|^2 = |E_{xs}|^2 + |E_{ys}|^2$. Equivalently $S_{\text{avg}} = v_p\langle w_e + w_m\rangle = v_p\cdot\frac{1}{2}\varepsilon E_0^2$: the stored energy is carried along at v_p.

AVERAGE POWER DENSITY OF A PLANE WAVE, LOSSY MEDIUM

$$\mathbf{S}_{\text{avg}} = \frac{E_0^2}{2|\eta|}\,e^{-2\alpha z}\cos\theta_\eta\,\hat{a}_z$$

Wave $E_0 e^{-\alpha z}\cos(\omega t - \beta z)\hat{a}_x$ with $\eta = |\eta|\angle\theta_\eta$, so $\mathbf{H} = (E_0/|\eta|)e^{-\alpha z}\cos(\omega t - \beta z - \theta_\eta)\hat{a}_y$. Power decays as $e^{-2\alpha z}$, twice the exponent of the field.

TOTAL POWER THROUGH A SURFACE

$$P_{\text{avg}} = \int_S \mathbf{S}_{\text{avg}}\cdot d\mathbf{S} \;=\; S_{\text{avg}}A \;\;(\text{uniform, normal to a flat area } A)$$

If the area is tilted by angle ψ between its normal and $\mathbf{S}_{\text{avg}}$, multiply by $\cos\psi$.

Plane waves: propagation, reflection, polarization and skin depth

PHASE CONSTANT (LOSSLESS)

$$\beta = \omega\sqrt{\mu\varepsilon} = \frac{\omega}{c}\sqrt{\mu_r\varepsilon_r} = \frac{2\pi}{\lambda}$$

In rad/m; λ is the wavelength in the medium.

PHASE VELOCITY AND WAVELENGTH (LOSSLESS)

$$v_p = \frac{\omega}{\beta} = \frac{1}{\sqrt{\mu\varepsilon}} = \frac{c}{\sqrt{\mu_r\varepsilon_r}}, \qquad \lambda = \frac{v_p}{f} = \frac{\lambda_0}{\sqrt{\mu_r\varepsilon_r}}$$

$\lambda_0 = c/f$ is the free-space wavelength. Refractive index $n = \sqrt{\mu_r\varepsilon_r} = c/v_p$. The frequency does not change when a wave enters a new medium; the wavelength does.

INTRINSIC IMPEDANCE (LOSSLESS)

$$\eta = \sqrt{\frac{\mu}{\varepsilon}} = \eta_0\sqrt{\frac{\mu_r}{\varepsilon_r}}, \qquad \eta_0 = \sqrt{\frac{\mu_0}{\varepsilon_0}} \approx 120\pi \approx 377\ \Omega$$

Ratio $|\mathbf{E}|/|\mathbf{H}|$ for a single travelling wave. For a non-magnetic dielectric $\eta = \eta_0/n$.

DIRECTION RELATIONS

$$\hat{a}_k = \hat{a}_E \times \hat{a}_H, \qquad \mathbf{H}_s = \frac{1}{\eta}\hat{a}_k \times \mathbf{E}_s, \qquad \mathbf{E}_s = -\eta\hat{a}_k \times \mathbf{H}_s$$

Use these to get $\mathbf{H}$ from $\mathbf{E}$ (or back) for any propagation direction, including a reflected wave travelling in $-z$. They also hold in a lossy medium, with complex η.

PLANE WAVE IN AN ARBITRARY DIRECTION

$$\mathbf{E}_s = \mathbf{E}_0 e^{-j\mathbf{k}\cdot\mathbf{r}}, \qquad \mathbf{k} = k_x\hat{a}_x + k_y\hat{a}_y + k_z\hat{a}_z = \beta\hat{a}_k, \qquad \mathbf{k}\cdot\mathbf{E}_0 = 0, \qquad \mathbf{H}_s = \frac{\mathbf{k}\times\mathbf{E}_s}{\omega\mu}$$

Lossless medium; $\mathbf{r} = x\hat{a}_x + y\hat{a}_y + z\hat{a}_z$ and $\beta = \sqrt{k_x^2 + k_y^2 + k_z^2} = \omega\sqrt{\mu\varepsilon}$. Time form (real $\mathbf{E}_0$): $\mathbf{E}_0\cos(\omega t - \mathbf{k}\cdot\mathbf{r})$. Example: $\cos(\omega t - 3x - 4y)$ with x, y in metres has $\beta = 5$ rad/m and $\hat{a}_k = 0.6\hat{a}_x + 0.8\hat{a}_y$. The wavelength measured along one axis, e.g. $2\pi/k_x$, is never shorter than $\lambda = 2\pi/\beta$.

TIME-AVERAGE POWER DENSITY (LOSSLESS)

$$\mathbf{S}_{\text{avg}} = \frac{1}{2}\,\text{Re}\,(\mathbf{E}_s \times \mathbf{H}_s^*) = \frac{|E_0|^2}{2\eta}\hat{a}_k$$

In W/m². Power through an area A normal to $\hat{a}_k$ is $S_{\text{avg}}A$. With RMS amplitudes, drop the factor 1/2.

PROPAGATION CONSTANT

$$\gamma = \alpha + j\beta = \sqrt{j\omega\mu\,(\sigma + j\omega\varepsilon)}$$

Take the root with $\alpha \geq 0$ and $\beta \geq 0$ for a wave travelling in $+z$ as $e^{-\gamma z}$.

EXACT ATTENUATION AND PHASE CONSTANTS

$$\alpha = \omega\sqrt{\frac{\mu\varepsilon}{2}\left[\sqrt{1 + \left(\frac{\sigma}{\omega\varepsilon}\right)^2} - 1\right]}, \qquad \beta = \omega\sqrt{\frac{\mu\varepsilon}{2}\left[\sqrt{1 + \left(\frac{\sigma}{\omega\varepsilon}\right)^2} + 1\right]}$$

Valid for any σ. Use it when the loss tangent is neither very small nor very large.

LOSS TANGENT AND COMPLEX PERMITTIVITY

$$\tan\theta = \frac{\sigma}{\omega\varepsilon} = \frac{\varepsilon''}{\varepsilon'}, \qquad \varepsilon_c = \varepsilon' - j\varepsilon'' = \varepsilon - j\frac{\sigma}{\omega}$$

Equals $|\mathbf{J}_c|/|\mathbf{J}_d|$. Writing $\varepsilon' = \varepsilon$ and $\varepsilon'' = \sigma/\omega$ assumes all the loss is conduction loss. Good dielectric: $\tan\theta \ll 1$; good conductor: $\tan\theta \gg 1$. The same material can be either, depending on frequency.

INTRINSIC IMPEDANCE (LOSSY)

$$\eta = \sqrt{\frac{j\omega\mu}{\sigma + j\omega\varepsilon}} = |\eta|\angle\theta_\eta, \qquad |\eta| = \frac{\sqrt{\mu/\varepsilon}}{\left[1 + \left(\frac{\sigma}{\omega\varepsilon}\right)^2\right]^{1/4}}, \qquad \tan 2\theta_\eta = \frac{\sigma}{\omega\varepsilon}$$

$0 \leq \theta_\eta \leq 45°$ and $\theta = 2\theta_\eta$. $\mathbf{E}$ leads $\mathbf{H}$ in time by θ_η.

POWER DECAY AND ATTENUATION IN DB

$$\mathbf{S}_{\text{avg}} = \frac{E_0^2}{2|\eta|}e^{-2\alpha z}\cos\theta_\eta\hat{a}_z, \qquad \text{attenuation (dB)} = 20\log_{10}e^{\alpha z} = 8.686\alpha z$$

Power decays at 2α, twice the field rate, but the dB figure is the same for field and power: 1 Np = 8.686 dB.

GOOD DIELECTRIC APPROXIMATIONS

$$\alpha \approx \frac{\sigma}{2}\sqrt{\frac{\mu}{\varepsilon}}, \qquad \beta \approx \omega\sqrt{\mu\varepsilon}, \qquad \eta \approx \sqrt{\frac{\mu}{\varepsilon}}\left(1 + j\frac{\sigma}{2\omega\varepsilon}\right)$$

For $\sigma/(\omega\varepsilon) \ll 1$, to first order in the loss tangent. If σ and ε do not depend on frequency, α is then independent of frequency; β, v_p and η are essentially the lossless values.

GOOD CONDUCTOR APPROXIMATIONS

$$\alpha \approx \beta \approx \sqrt{\pi f \mu \sigma} = \sqrt{\frac{\omega\mu\sigma}{2}}, \qquad \eta \approx \sqrt{\frac{\omega\mu}{\sigma}}\angle 45^\circ = \frac{1+j}{\sigma\delta}$$

For $\sigma/(\omega\varepsilon) \gg 1$. $|\eta|$ is tiny (a small fraction of an ohm for metals at radio frequencies), so a good conductor looks almost like a short circuit to an incident wave.

SKIN DEPTH

$$\delta = \frac{1}{\alpha}, \qquad \delta_{\text{good conductor}} = \frac{1}{\sqrt{\pi f \mu \sigma}}, \qquad \lambda = 2\pi\delta, \qquad v_p = \omega\delta$$

Distance over which the field amplitude falls to $1/e \approx 0.368$ of its value (power to $1/e^2 \approx 0.135$). In a good conductor $\delta \propto 1/\sqrt{f}$; the last two results hold only there.

SURFACE RESISTANCE AND AC RESISTANCE OF A WIRE

$$R_s = \mathrm{Re}(\eta) = \frac{1}{\sigma\delta} = \sqrt{\frac{\pi f \mu}{\sigma}}, \qquad R_{\text{ac}} \approx \frac{\ell}{\sigma(2\pi a)\delta} = \frac{\ell R_s}{2\pi a}$$

Good conductor, with thickness (or wire radius a) much greater than δ: the current is treated as flowing uniformly in a surface layer of depth δ. ℓ is the wire length.

PHASE AND GROUP VELOCITY

$$v_p = \frac{\omega}{\beta}, \qquad v_g = \frac{d\omega}{d\beta} = \frac{1}{d\beta/d\omega}$$

Evaluate $d\beta/d\omega$ at the carrier frequency.

ENVELOPE VELOCITY OF TWO CLOSE FREQUENCIES

$$v_g \approx \frac{\omega_1 - \omega_2}{\beta_1 - \beta_2}$$

For $\cos(\omega_1 t - \beta_1 z) + \cos(\omega_2 t - \beta_2 z)$ the envelope $\cos\left(\frac{\Delta\omega}{2}t - \frac{\Delta\beta}{2}z\right)$ moves at $\Delta\omega/\Delta\beta$, while the carrier moves at $(\omega_1+\omega_2)/(\beta_1+\beta_2)$.

PRODUCT AND RATIO RESULTS

$$v_p v_g = c^2 \quad \text{(plasma)}, \qquad v_g = 2v_p \quad \text{(good conductor)}$$

$v_p v_g = c^2$ holds whenever $\beta = \sqrt{\omega^2 - \omega_0^2}/c$ (plasma, air-filled waveguide above cut-off). In a good conductor $\beta = \sqrt{\omega\mu\sigma/2}$ gives $d\beta/d\omega = \beta/(2\omega)$.

CIRCULAR POLARISATION PHASOR SHORTCUT (IEEE)

$$\text{With } e^{-j\beta z}: \quad (\hat{a}_x - j\hat{a}_y) \to \text{RHCP}, \qquad (\hat{a}_x + j\hat{a}_y) \to \text{LHCP}$$

Reverse both for a wave travelling in $-z$ ($e^{+j\beta z}$). The optics convention names handedness the opposite way, so use the definition the question states.

REFLECTION AND TRANSMISSION COEFFICIENTS (E-FIELD)

$$\Gamma = \frac{E_{r0}}{E_{i0}} = \frac{\eta_2 - \eta_1}{\eta_2 + \eta_1}, \qquad \tau = \frac{E_{t0}}{E_{i0}} = \frac{2\eta_2}{\eta_2 + \eta_1}, \qquad \tau = 1 + \Gamma$$

For real η: $-1 \le \Gamma \le 1$ and $0 \le \tau \le 2$. Perfect conductor: $\eta_2 = 0$, so $\Gamma = -1$ and $\tau = 0$.

NON-MAGNETIC DIELECTRICS IN TERMS OF REFRACTIVE INDEX

$$\Gamma = \frac{n_1 - n_2}{n_1 + n_2}, \qquad \tau = \frac{2n_1}{n_1 + n_2} \qquad (\mu_1 = \mu_2 = \mu_0)$$

Follows from $\eta = \eta_0/n$. Entering a denser medium ($n_2 > n_1$) gives $\Gamma < 0$.

REFLECTED AND TRANSMITTED POWER FRACTIONS

$$R = |\Gamma|^2, \qquad T = 1 - |\Gamma|^2 = \frac{4\eta_1 \,\mathrm{Re}(\eta_2)}{|\eta_1 + \eta_2|^2} \qquad (\eta_1 \text{ real})$$

These need only medium 1 to be lossless; η_2 may be complex. For a good conductor $T \approx 4R_s/\eta_1$, a tiny fraction. When both impedances are real, $T = (\eta_1/\eta_2)|\tau|^2$; $T \neq |\tau|^2$ because the impedances on the two sides differ.

STANDING WAVE RATIO IN MEDIUM 1

$$s = \frac{|E_1|_{\max}}{|E_1|_{\min}} = \frac{1 + |\Gamma|}{1 - |\Gamma|}, \qquad |\Gamma| = \frac{s - 1}{s + 1}$$

$1 \le s \le \infty$. With lossless medium 1, $|E_1|_{\max} = (1 + |\Gamma|)E_{i0}$ and $|E_1|_{\min} = (1 - |\Gamma|)E_{i0}$.

WHERE THE MAXIMA AND MINIMA SIT (REAL Γ)

$$\Gamma > 0: \quad |E_1|_{\max} \text{ at } z = -\frac{m\lambda_1}{2}; \qquad \Gamma < 0: \quad |E_1|_{\min} \text{ at } z = -\frac{m\lambda_1}{2}, \qquad m = 0, 1, 2, \ldots$$

Adjacent maximum and minimum are $\lambda_1/4$ apart. $\Gamma > 0$ when $\eta_2 > \eta_1$ (e.g. dielectric into air); $\Gamma < 0$ when $\eta_2 < \eta_1$ (e.g. air into a dielectric or conductor). $|\mathbf{H}_1|$ has its maxima where $|\mathbf{E}_1|$ has its minima. For complex $\Gamma = |\Gamma|\angle\theta_\Gamma$ (lossy medium 2, lossless medium 1), $|E_1|$ is maximum at $z = -(\theta_\Gamma + 2m\pi)/(2\beta_1)$, with the integers m chosen so that $z \le 0$, and minimum $\lambda_1/4$ from each maximum.

SLAB OF THICKNESS D BETWEEN TWO MEDIA (LOSSLESS SLAB)

$$\eta_{\text{in}} = \eta_2 \frac{\eta_3 + j\eta_2 \tan\beta_2 d}{\eta_2 + j\eta_3 \tan\beta_2 d}, \qquad \Gamma_{\text{front}} = \frac{\eta_{\text{in}} - \eta_1}{\eta_{\text{in}} + \eta_1}$$

Same form as a transmission-line input impedance. Quarter-wave slab ($d = \lambda_2/4$, or any odd multiple): $\eta_{\text{in}} = \eta_2^2/\eta_3$, so $\eta_2 = \sqrt{\eta_1\eta_3}$ removes the reflection, but only at the design frequency. Half-wave slab ($d = \lambda_2/2$): $\eta_{\text{in}} = \eta_3$, so the reflection is the same as if medium 3 met medium 1 directly; it is zero only when $\eta_3 = \eta_1$ (e.g. a dielectric sheet in air).

LAW OF REFLECTION AND SNELL'S LAW

$$\theta_r = \theta_i, \qquad \beta_1 \sin\theta_i = \beta_2 \sin\theta_t \quad \Leftrightarrow \quad n_1 \sin\theta_i = n_2 \sin\theta_t$$

$n = \sqrt{\mu_r\varepsilon_r}$. Both follow from phase matching: the phase constant along the boundary is the same for all three waves.

FRESNEL COEFFICIENTS, PERPENDICULAR (TE) POLARISATION

$$\Gamma_\perp = \frac{\eta_2 \cos\theta_i - \eta_1 \cos\theta_t}{\eta_2 \cos\theta_i + \eta_1 \cos\theta_t}, \qquad \tau_\perp = \frac{2\eta_2 \cos\theta_i}{\eta_2 \cos\theta_i + \eta_1 \cos\theta_t}, \qquad \tau_\perp = 1 + \Gamma_\perp$$

Electric-field ratios. At $\theta_i = 0$ they reduce to the normal-incidence Γ and τ.

FRESNEL COEFFICIENTS, PARALLEL (TM) POLARISATION

$$\Gamma_\parallel = \frac{\eta_2 \cos\theta_t - \eta_1 \cos\theta_i}{\eta_2 \cos\theta_t + \eta_1 \cos\theta_i}, \qquad \tau_\parallel = \frac{2\eta_2 \cos\theta_i}{\eta_2 \cos\theta_t + \eta_1 \cos\theta_i}, \qquad 1 + \Gamma_\parallel = \tau_\parallel \frac{\cos\theta_t}{\cos\theta_i}$$

Sign convention chosen so that $\Gamma_\parallel \to (\eta_2 - \eta_1)/(\eta_2 + \eta_1)$ at normal incidence; some texts define $\Gamma_\parallel$ with the opposite sign. $|\Gamma_\parallel|$ and the Brewster condition do not depend on that choice.

BREWSTER ANGLE (NON-MAGNETIC MEDIA)

$$\tan\theta_B = \frac{n_2}{n_1} = \sqrt{\frac{\varepsilon_2}{\varepsilon_1}}, \qquad \theta_B + \theta_t = 90^\circ$$

Equivalently $\sin\theta_B = \sqrt{\varepsilon_2/(\varepsilon_1 + \varepsilon_2)}$. The angle at which $\Gamma_\parallel = 0$. Between non-magnetic media there is no Brewster angle for perpendicular polarisation, so an unpolarised wave incident at θ_B reflects as a purely perpendicular-polarised wave (the polarising angle).

CRITICAL ANGLE

$$\sin\theta_c = \frac{n_2}{n_1} = \sqrt{\frac{\varepsilon_2}{\varepsilon_1}} \qquad (n_1 > n_2;\ \mu_1 = \mu_2)$$

Exists only going from the denser to the rarer medium. For $\theta_i \geq \theta_c$, $|\Gamma_\perp| = |\Gamma_\parallel| = 1$ (total internal reflection) for both polarisations, and the reflected wave picks up a phase shift that differs between the two.

Transmission lines, impedance matching, S-parameters and the Smith chart

TELEGRAPHER'S EQUATIONS (PHASOR FORM)

$$\frac{dV}{dz} = -(R + j\omega L)\, I, \qquad \frac{dI}{dz} = -(G + j\omega C)\, V, \qquad \frac{d^2V}{dz^2} = \gamma^2 V$$

Sinusoidal steady state with $e^{j\omega t}$; the third equation follows by differentiating the first and substituting the second.

GENERAL SOLUTION

$$V(z) = V_0^+ e^{-\gamma z} + V_0^- e^{\gamma z}, \qquad I(z) = \frac{V_0^+}{Z_0}\, e^{-\gamma z} - \frac{V_0^-}{Z_0}\, e^{\gamma z}$$

V_0^+ is the wave travelling toward the load and V_0^- the reflected wave. The minus sign in $I(z)$ matters: the reflected wave's current flows back toward the generator.

PROPAGATION CONSTANT AND CHARACTERISTIC IMPEDANCE

$$\gamma = \alpha + j\beta = \sqrt{(R + j\omega L)(G + j\omega C)}, \qquad Z_0 = \sqrt{\frac{R + j\omega L}{G + j\omega C}}$$

α in Np/m (1 Np = 8.686 dB), β in rad/m. $Z_0 = V^+/I^+$ for ONE travelling wave; it is not V/I at a point on a mismatched line. Inverting, $R + j\omega L = \gamma Z_0$ and $G + j\omega C = \gamma / Z_0$ recover the line constants from measured γ and Z_0.

LOSSLESS LINE ($R = G = 0$)

$$Z_0 = \sqrt{\frac{L}{C}}, \quad \alpha = 0, \quad \beta = \omega\sqrt{LC}, \quad v_p = \frac{\omega}{\beta} = \frac{1}{\sqrt{LC}}, \quad \lambda = \frac{2\pi}{\beta} = \frac{v_p}{f}$$

Z_0 is real and, with L and C constant, independent of frequency. On any TEM line whose dielectric is homogeneous (coaxial, two-wire, parallel-plate) $LC = \mu\varepsilon$; for a non-magnetic dielectric of relative permittivity ε_r this gives $v_p = c/\sqrt{\varepsilon_r}$, so the wavelength on the line is shorter than in free space by the factor $\sqrt{\varepsilon_r}$.

DISTORTIONLESS LINE (HEAVISIDE CONDITION)

$$\frac{R}{L} = \frac{G}{C} \ \Rightarrow\ \alpha = \sqrt{RG}, \quad \beta = \omega\sqrt{LC}, \quad Z_0 = \sqrt{\frac{L}{C}} = \sqrt{\frac{R}{G}}$$

The line is lossy, but (taking R, L, G, C as constants) α and v_p do not depend on frequency and Z_0 is real, so a signal is attenuated and delayed without changing shape.

LOW-LOSS APPROXIMATION

$$R \ll \omega L,\ G \ll \omega C: \quad \alpha \approx \frac{R}{2Z_0} + \frac{GZ_0}{2}, \quad \beta \approx \omega\sqrt{LC}, \quad Z_0 \approx \sqrt{\frac{L}{C}}$$

The first term of α is conductor loss, the second dielectric loss.

LOAD REFLECTION COEFFICIENT

$$\Gamma_L = \frac{V_0^-}{V_0^+} = \frac{Z_L - Z_0}{Z_L + Z_0} = \frac{z_L - 1}{z_L + 1}, \qquad z_L = \frac{Z_L}{Z_0} = \frac{1 + \Gamma_L}{1 - \Gamma_L}$$

Voltage reflection coefficient; $|\Gamma_L| \leq 1$ for any passive load ($\mathrm{Re}\, Z_L \geq 0$) on a line with real Z_0. The current reflection coefficient is $-\Gamma_L$.

TRANSMISSION COEFFICIENT AT A JUNCTION

$$\tau = 1 + \Gamma_L = \frac{2Z_L}{Z_L + Z_0}, \qquad \text{line } Z_{01} \text{ into line } Z_{02}: \ \Gamma = \frac{Z_{02} - Z_{01}}{Z_{02} + Z_{01}}$$

Voltage is continuous at the load or junction, so transmitted voltage = incident + reflected. For a junction the 'load' is the second line's Z_{02} (assumed matched beyond). τ can exceed 1, but power still balances: for real Z_{01}, Z_{02} the fraction of incident power transmitted is $|\tau|^2 Z_{01}/Z_{02} = 1 - |\Gamma|^2$.

REFLECTION COEFFICIENT ALONG THE LINE

$$\Gamma(d) = \Gamma_L\, e^{-2\gamma d}, \qquad \text{lossless: } \Gamma(d) = |\Gamma_L|\, e^{j(\theta_L - 2\beta d)}$$

$\theta_L = \angle\Gamma_L$ and d is measured from the load toward the generator. On a lossy line $|\Gamma|$ shrinks as $e^{-2\alpha d}$ toward the generator.

INPUT IMPEDANCE OF A LOSSLESS LINE

$$Z_{in}(d) = Z_0\,\frac{Z_L + jZ_0\tan\beta d}{Z_0 + jZ_L\tan\beta d} = Z_0\,\frac{1+\Gamma(d)}{1-\Gamma(d)}$$

Every input-impedance result in the next section is a special case of this formula. Z_{in} repeats every $\lambda/2$, because $\tan\beta d$ has period π.

VOLTAGE STANDING WAVE RATIO

$$S = \frac{|V|_{max}}{|V|_{min}} = \frac{1+|\Gamma|}{1-|\Gamma|}, \qquad |\Gamma| = \frac{S-1}{S+1}, \qquad 1 \le S \le \infty$$

Lossless line: $|V|_{max} = |V_0^+|(1+|\Gamma|)$ and $|V|_{min} = |V_0^+|(1-|\Gamma|)$. The current maxima sit at the voltage minima.

RETURN LOSS, DELIVERED POWER AND MISMATCH LOSS

$$RL = -20\log_{10}|\Gamma|\ \text{dB}, \qquad P_L = \frac{|V_0^+|^2}{2Z_0}\left(1-|\Gamma|^2\right), \qquad ML = -10\log_{10}\left(1-|\Gamma|^2\right)\ \text{dB}$$

Lossless line, V_0^+ a peak phasor. $|\Gamma|^2$ is the fraction of incident power reflected. The same delivered power is $P_L = |V|_{max}|V|_{min}/(2Z_0)$. Return loss is quoted here as a positive number for a passive load (a matched load has $RL = \infty$); some texts attach a minus sign, so read the question's convention.

LINE DRIVEN BY A GENERATOR

$$I_{in} = \frac{V_g}{Z_g + Z_{in}}, \qquad V_{in} = V_g\,\frac{Z_{in}}{Z_g + Z_{in}}, \qquad P_{in} = \frac{1}{2}|I_{in}|^2\,\mathrm{Re}\,Z_{in}$$

V_g is the source's open-circuit voltage (peak phasor) and Z_g its internal impedance; Z_{in} is the input impedance of the line together with its load. On a lossless line all of P_{in} reaches the load. For a given source, P_{in} is largest when $Z_{in} = Z_g^*$ (conjugate matching); when $Z_g = Z_0$ is real, that happens exactly when the load is matched, $Z_L = Z_0$.

STEP RESPONSE OF A LOSSLESS LINE (RESISTIVE SOURCE AND LOAD)

$$V_1^+ = V_g\,\frac{Z_0}{R_g + Z_0}, \qquad \Gamma_g = \frac{R_g - Z_0}{R_g + Z_0}, \qquad \Gamma_L = \frac{R_L - Z_0}{R_L + Z_0}$$

$$V(t\to\infty) = V_g\,\frac{R_L}{R_g + R_L}$$

A step of height V_g is applied at $t = 0$ through a source resistance R_g to a line of one-way delay $T = \ell/v_p$. The line first looks like Z_0, so it launches V_1^+. At $t = T$ this reaches the load, which reflects $\Gamma_L V_1^+$, so the load voltage jumps to $V_1^+(1+\Gamma_L)$; at $t = 2T$ the reflection reaches the source and is reflected again with Γ_g, and so on. The voltage at any point is the sum of every wave that has passed it so far (a bounce diagram). The final value is the DC voltage divider, because at DC a lossless line is just a pair of wires.

POSITIONS OF VOLTAGE MAXIMA AND MINIMA

$$d_{max} = \frac{\theta_L}{4\pi}\lambda + n\frac{\lambda}{2}, \qquad d_{min} = \frac{\theta_L + \pi}{4\pi}\lambda + n\frac{\lambda}{2}, \qquad Z(d_{max}) = SZ_0, \quad Z(d_{min}) = \frac{Z_0}{S}$$

Lossless line, θ_L in radians, n an integer; the first maximum (or minimum) is the smallest $d \geq 0$ the formula gives. Successive maxima (or minima) are $\lambda/2$ apart and a maximum is $\lambda/4$ from the next minimum. The line impedance is purely resistive at both.

SHORT-CIRCUITED LINE

$$Z_{sc} = jZ_0 \tan \beta\ell$$

Purely reactive. Inductive for $0 < \ell < \lambda/4$ (with $Z_{sc} = jZ_0$ at $\ell = \lambda/8$); an open circuit at $\ell = \lambda/4$, near which it behaves like a parallel resonant circuit; capacitive for $\lambda/4 < \ell < \lambda/2$; a short circuit again at $\ell = \lambda/2$.

OPEN-CIRCUITED LINE

$$Z_{oc} = -jZ_0 \cot \beta\ell$$

Purely reactive. Capacitive for $0 < \ell < \lambda/4$ (with $Z_{oc} = -jZ_0$ at $\ell = \lambda/8$); a short circuit at $\ell = \lambda/4$, near which it behaves like a series resonant circuit; inductive for $\lambda/4 < \ell < \lambda/2$; an open circuit again at $\ell = \lambda/2$.

LINE CONSTANTS FROM OPEN- AND SHORT-CIRCUIT MEASUREMENTS

$$Z_0 = \sqrt{Z_{oc}\, Z_{sc}}, \qquad \tanh \gamma\ell = \frac{Z_{sc}}{Z_0}, \qquad \tanh^2 \gamma\ell = \frac{Z_{sc}}{Z_{oc}}$$

Valid for lossy lines too, since $Z_{sc} = Z_0 \tanh \gamma\ell$ and $Z_{oc} = Z_0 \coth \gamma\ell$; take the root of Z_0 with $\mathrm{Re}\, Z_0 > 0$. For a lossless line write $Z_{sc} = jX_{sc}$ and $Z_{oc} = jX_{oc}$: then $Z_0 = \sqrt{-X_{sc}X_{oc}}$ and $\tan \beta\ell = X_{sc}/Z_0$, which fixes ℓ to within a multiple of $\lambda/2$. (Taking $\tan \beta\ell = \sqrt{-Z_{sc}/Z_{oc}}$ loses the sign: a $3\lambda/8$ line would be read as $\lambda/8$.)

QUARTER-WAVE LINE (IMPEDANCE INVERTER)

$$\ell = \frac{\lambda}{4}\ (\text{or an odd multiple}): \qquad Z_{in} = \frac{Z_0^2}{Z_L}, \qquad z_{in} = \frac{1}{z_L}$$

A short becomes an open, an open becomes a short, and an inductive load looks capacitive (and vice versa).

HALF-WAVE LINE

$$\ell = n\frac{\lambda}{2}: \qquad Z_{in} = Z_L$$

For any Z_0: at a frequency where the section is a whole number of half-wavelengths long, it repeats the load impedance.

QUARTER-WAVE TRANSFORMER (REAL LOAD)

$$Z_T = \sqrt{Z_0\, R_L}$$

A $\lambda/4$ section of characteristic impedance Z_T makes a real load R_L look like Z_0. Exact only at the design frequency (where the section is $\lambda/4$ long) and at odd multiples of it.

QUARTER-WAVE TRANSFORMER FOR A COMPLEX LOAD

$$Z_T = Z_0\sqrt{S}\ \text{ inserted at a voltage maximum}, \qquad Z_T = \frac{Z_0}{\sqrt{S}}\ \text{ inserted at a voltage minimum}$$

First move from the load to a point where the line impedance is real (SZ_0 at d_{max}, Z_0/S at d_{min}), cut the line there and insert the $\lambda/4$ section: $Z_T = \sqrt{Z_0 \cdot SZ_0}$ or $\sqrt{Z_0 \cdot Z_0/S}$.

ADMITTANCE TRANSFORMATION ALONG A LOSSLESS LINE

$$y(d) = \frac{y_L + j\tan \beta d}{1 + j\, y_L \tan \beta d}, \qquad y = \frac{Y}{Y_0} = \frac{Z_0}{Z}$$

Same form as the impedance formula, written in normalised admittances. Use it whenever elements are connected in shunt.

SHUNT-STUB POSITION FOR A RESISTIVE LOAD

$$z_L = r \ (\text{real},\ r \neq 1): \quad \tan \beta d = \pm\sqrt{r}, \qquad y(d) = 1 \pm j\,\frac{r-1}{\sqrt{r}}$$

Lossless line; the upper signs go together. Obtained by setting $\operatorname{Re} y(d) = 1$ with $y_L = 1/r$. Take βd in $(0, \pi)$, adding π to a negative angle.

THE MAPPING BEHIND THE CHART

$$\Gamma = \Gamma_r + j\Gamma_i = \frac{z-1}{z+1}, \qquad z = r + jx = \frac{1+\Gamma}{1-\Gamma}$$

The centre of the chart is $\Gamma = 0$, the matched point $z = 1$.

ROTATION FOR A MOVE ALONG THE LINE

$$\Delta\phi = 2\beta d = 4\pi\frac{d}{\lambda}\ \text{rad} = 720^\circ \times \frac{d}{\lambda}$$

With the $e^{j\omega t}$ convention: clockwise when moving toward the generator, counter-clockwise toward the load. One full turn is $\lambda/2$, so the rim scales run from 0 to 0.5 wavelengths.

DEFINITION OF THE SCATTERING MATRIX

$$\begin{bmatrix} b_1 \\ b_2 \end{bmatrix} = \begin{bmatrix} S_{11} & S_{12} \\ S_{21} & S_{22} \end{bmatrix} \begin{bmatrix} a_1 \\ a_2 \end{bmatrix}, \qquad S_{ij} = \left.\frac{b_i}{a_j}\right|_{a_k=0,\ k\neq j}$$

S_{11} is the reflection coefficient at port 1 with port 2 terminated in Z_0 (matched, not shorted); S_{21} is the transmission coefficient from port 1 to port 2 under the same condition.

INPUT REFLECTION COEFFICIENT WITH AN ARBITRARY LOAD

$$\Gamma_{in} = S_{11} + \frac{S_{12}S_{21}\Gamma_L}{1 - S_{22}\Gamma_L}$$

Γ_L is the reflection coefficient of the load on port 2, referred to Z_0. $\Gamma_{in} = S_{11}$ only when $\Gamma_L = 0$. Looking back into port 2 with a source of reflection coefficient Γ_s on port 1, $\Gamma_{out} = S_{22} + S_{12}S_{21}\Gamma_s/(1 - S_{11}\Gamma_s)$.

RETURN LOSS, INSERTION LOSS AND INPUT VSWR

$$RL_1 = -20\log_{10}|S_{11}|\ \text{dB}, \qquad IL = -20\log_{10}|S_{21}|\ \text{dB}, \qquad \text{VSWR}_1 = \frac{1+|S_{11}|}{1-|S_{11}|}$$

All with port 2 matched. $|S_{21}|^2$ is the fraction of the incident power delivered to the matched load on port 2.

RECIPROCITY

$$S_{12} = S_{21}, \quad \text{i.e. } [S] = [S]^T$$

Holds for linear networks built only from reciprocal elements (line sections, R, L, C, coupled coils and transformers), with the same real reference impedance at both ports. Fails for magnetically biased ferrite devices (isolators, circulators) and for active circuits such as amplifiers.

LOSSLESSNESS (UNITARY S-MATRIX)

$$[S]^{*T}[S] = [I] \ \Rightarrow\ |S_{11}|^2 + |S_{21}|^2 = 1, \quad |S_{12}|^2 + |S_{22}|^2 = 1, \quad S_{11}^*S_{12} + S_{21}^*S_{22} = 0$$

Each column carries unit power and different columns are orthogonal (equivalently, $[S][S]^{*T} = [I]$, so the rows are too). For a lossless two-port this forces $|S_{11}| = |S_{22}|$ and $|S_{12}| = |S_{21}|$. Any passive network has $|S_{11}|^2 + |S_{21}|^2 \leq 1$; the shortfall is the fraction of the power incident on port 1 that is dissipated inside the network (port 2 matched).

Waveguides and optical fibres

PROPAGATION CONSTANT OF A WAVEGUIDE MODE

$$\beta = \sqrt{k^2 - k_c^2} = \omega\sqrt{\mu\varepsilon}\sqrt{1 - \left(\frac{f_c}{f}\right)^2}$$

k_c is the mode's cut-off wavenumber, set by the guide's shape and size, and $f_c = k_c u/(2\pi)$. β is real (the mode propagates) only when $f > f_c$.

CUT-OFF FREQUENCY, RECTANGULAR GUIDE (TE AND TM)

$$f_{c,mn} = \frac{1}{2\sqrt{\mu\varepsilon}}\sqrt{\left(\frac{m}{a}\right)^2 + \left(\frac{n}{b}\right)^2} = \frac{c}{2\sqrt{\mu_r\varepsilon_r}}\sqrt{\left(\frac{m}{a}\right)^2 + \left(\frac{n}{b}\right)^2}$$

Same formula for TE and TM. Allowed indices: TE_{mn} with $m, n = 0, 1, 2, \ldots$ but not both zero, and TM_{mn} with $m, n \geq 1$. A filling with $\mu_r = 1$ divides every air-filled cut-off by $\sqrt{\varepsilon_r}$.

CUT-OFF WAVENUMBER AND CUT-OFF WAVELENGTH

$$k_c = \sqrt{\left(\frac{m\pi}{a}\right)^2 + \left(\frac{n\pi}{b}\right)^2}, \quad \lambda_c = \frac{2\pi}{k_c} = \frac{2}{\sqrt{(m/a)^2 + (n/b)^2}}$$

λ_c depends only on the geometry, not on the filling. A mode propagates when the wavelength in the filling, $\lambda = u/f$, is shorter than its λ_c.

DOMINANT TE10 MODE

$$f_{c,10} = \frac{c}{2a\sqrt{\mu_r\varepsilon_r}}, \quad \lambda_{c,10} = 2a$$

When $a > b$, TE_{10} has the lowest cut-off of all modes, so it is the dominant mode. Its cut-off does not depend on b. In a square guide ($a = b$), TE_{01} has the same cut-off.

GUIDE WAVELENGTH

$$\lambda_g = \frac{2\pi}{\beta} = \frac{\lambda}{\sqrt{1 - (f_c/f)^2}} = \frac{\lambda}{\sqrt{1 - (\lambda/\lambda_c)^2}}$$

$\lambda = u/f$ is the wavelength in the unbounded filling, and $\lambda_g > \lambda$ for every propagating mode. For TE_{10}, $\lambda_c = 2a$, so $\lambda_g = \lambda/\sqrt{1 - (\lambda/2a)^2}$.

WAVELENGTH RELATION

$$\frac{1}{\lambda^2} = \frac{1}{\lambda_g^2} + \frac{1}{\lambda_c^2}$$

λ is the wavelength in the filling. This says the same thing as $k^2 = \beta^2 + k_c^2$. It is the quickest route when two of the three wavelengths are given.

PHASE AND GROUP VELOCITY

$$v_p = \frac{\omega}{\beta} = \frac{u}{\sqrt{1 - (f_c/f)^2}}, \quad v_g = \frac{d\omega}{d\beta} = u\sqrt{1 - (f_c/f)^2}, \quad v_p v_g = u^2$$

$v_p > u$ is the speed of the phase pattern and carries no energy or information. $v_g < u$ is the speed of energy in a lossless guide. Also, $v_p = f\lambda_g$.

WAVE IMPEDANCE OF TE AND TM MODES

$$Z_{TE} = \frac{\omega\mu}{\beta} = \frac{\eta}{\sqrt{1 - (f_c/f)^2}}, \quad Z_{TM} = \frac{\beta}{\omega\varepsilon} = \eta\sqrt{1 - (f_c/f)^2}, \quad Z_{TE}Z_{TM} = \eta^2$$

This is the ratio of the mode's transverse E to its transverse H (for TE_{10}, $Z_{TE} = E_y/(-H_x)$). Above cut-off, $Z_{TE} > \eta > Z_{TM}$. Equivalently, $Z_{TE} = \eta\lambda_g/\lambda$ and $Z_{TM} = \eta\lambda/\lambda_g$.

ATTENUATION BELOW CUT-OFF (EVANESCENT MODE)

$$\alpha = \sqrt{k_c^2 - k^2} = \frac{2\pi}{\lambda_c}\sqrt{1 - \left(\frac{f}{f_c}\right)^2} \quad (\mathrm{Np/m})$$

For $f < f_c$ the fields vary as $e^{-\alpha z}$, with no phase shift. $Z_{TE} = j\omega\mu/\alpha$ becomes inductive and $Z_{TM} = -j\alpha/(\omega\varepsilon)$ becomes capacitive, so no average power flows. Well below cut-off, $\alpha \approx 2\pi/\lambda_c$.

POWER CARRIED BY THE TE10 MODE

$$P = \frac{E_0^2 ab}{4Z_{TE}}$$

Time-average power in a lossless guide. E_0 is the peak phasor amplitude of E_y on the centre line $x = a/2$, and Z_{TE} is the TE_{10} wave impedance.

CUT-OFF FREQUENCY AND WAVELENGTH, CIRCULAR GUIDE

$$f_{c,\mathrm{TE}_{nm}} = \frac{p'_{nm}}{2\pi a\sqrt{\mu\varepsilon}}, \quad f_{c,\mathrm{TM}_{nm}} = \frac{p_{nm}}{2\pi a\sqrt{\mu\varepsilon}}, \quad \lambda_c = \frac{2\pi a}{p'_{nm} \text{ or } p_{nm}}$$

a is the RADIUS. p_{nm} is the m-th non-zero root of $J_n(x)$ (TM), and p'_{nm} is the m-th non-zero root of $J'_n(x)$ (TE). The β, λ_g, v_p, v_g and Z_{TE}/Z_{TM} formulas from the previous section apply unchanged with this f_c.

DOMINANT TE11 MODE OF A CIRCULAR GUIDE

$$f_{c,11} = \frac{1.841c}{2\pi a\sqrt{\mu_r\varepsilon_r}} \approx \frac{0.293c}{a\sqrt{\mu_r\varepsilon_r}}, \quad \lambda_{c,11} = 3.41a$$

Air-filled: $f_{c,11} \approx 8.79/a$ GHz, with a in cm. Only TE_{11} propagates for $1.841 < ka < 2.405$, that is, up to the TM_{01} cut-off, a frequency ratio of about 1.31.

CRITICAL ANGLE AT THE CORE–CLADDING BOUNDARY

$$\sin\theta_c = \frac{n_2}{n_1}$$

θ_c is measured from the NORMAL to the boundary, inside the core. Rays that meet the boundary at $\phi > \theta_c$ are totally reflected and stay guided; $\phi = \theta_c$ is the limiting ray.

NUMERICAL APERTURE

$$\mathrm{NA} = n_0 \sin\theta_a = \sqrt{n_1^2 - n_2^2} = n_1\sqrt{2\Delta}, \quad \Delta = \frac{n_1^2 - n_2^2}{2n_1^2} \approx \frac{n_1 - n_2}{n_1}$$

Δ is the relative index difference. With the first form of Δ, $\mathrm{NA} = n_1\sqrt{2\Delta}$ exactly. The form $(n_1 - n_2)/n_1$ is the usual approximation for $\Delta \ll 1$. NA depends only on n_1 and n_2, not on the launch medium.

ACCEPTANCE ANGLE

$$\theta_a = \sin^{-1}\left(\frac{\mathrm{NA}}{n_0}\right), \quad \text{full acceptance cone} = 2\theta_a$$

θ_a is a half-angle measured from the fibre AXIS, outside the fibre. Rays inside this cone are guided. For the same fibre, a denser launch medium (larger n_0) gives a smaller θ_a. The formula needs $\mathrm{NA} \le n_0$; if $\mathrm{NA} > n_0$, every ray entering the end face is guided. This is the acceptance angle for meridional rays; skew rays, which never cross the axis, can be guided at somewhat larger launch angles.

NORMALISED FREQUENCY (V-NUMBER)

$$V = \frac{2\pi a}{\lambda_0}\mathrm{NA} = \frac{2\pi a}{\lambda_0}\sqrt{n_1^2 - n_2^2} = \frac{2\pi a}{\lambda_0}n_1\sqrt{2\Delta}$$

a is the core RADIUS and λ_0 is the free-space wavelength. V grows with core size and NA and falls as the wavelength rises.

SINGLE-MODE CONDITION (STEP-INDEX FIBRE)

$$V < 2.405 \quad \Leftrightarrow \quad \lambda_0 > \lambda_c = \frac{2\pi a \mathrm{NA}}{2.405} \quad \Leftrightarrow \quad a < \frac{2.405\lambda_0}{2\pi \mathrm{NA}}$$

2.405 is the first zero of J_0. The fundamental mode (LP_{01}, also called HE_{11}) has no cut-off, and below $V = 2.405$ it is the only guided mode (in two orthogonal polarisations). Here λ_c is the fibre's cut-off wavelength: the fibre is single-mode at every longer wavelength.

APPROXIMATE NUMBER OF GUIDED MODES

$$M \approx \frac{V^2}{2} \quad \text{(step-index)}, \qquad M \approx \frac{V^2}{4} \quad \text{(parabolic graded-index)}$$

These are the usual large-V estimates ($V \gg 1$), counting both polarisations. Use them only for multimode fibres. Both are cases of $M \approx \frac{g}{g+2} \cdot \frac{V^2}{2}$ for the power-law profile above.

MODAL (INTERMODAL) DELAY, STEP-INDEX FIBRE, RAY MODEL

$$\Delta\tau = \frac{Ln_1}{c}\left(\frac{n_1}{n_2} - 1\right) = \frac{Ln_1(n_1 - n_2)}{cn_2} \approx \frac{Ln_1\Delta}{c} \approx \frac{L\mathrm{NA}^2}{2n_1 c}$$

The difference in arrival time over length L between the axial ray (path L) and the steepest guided ray, at the critical angle (path Ln_1/n_2), both travelling at c/n_1. It is proportional to L and, for $\Delta \ll 1$, to Δ.

FIBRE LOSS IN DECIBELS

$$\alpha_{\mathrm{dB}} = \frac{10}{L}\log_{10}\left(\frac{P_{in}}{P_{out}}\right) \quad (\mathrm{dB/km, \text{ with } } L \text{ in km}), \qquad P_{out} = P_{in} \times 10^{-\alpha_{\mathrm{dB}} L/10}$$

Losses in dB add along a link: a fibre of α_{dB} dB/km loses $\alpha_{\mathrm{dB}}L$ dB over L km. A loss of 3 dB leaves about half the power, and 10 dB leaves one tenth.

Antennas and antenna arrays

FAR-FIELD DISTANCE

$$r \geq \frac{2D^2}{\lambda}, \qquad r \gg D, \qquad r \gg \lambda$$

D = largest dimension of the antenna (a length here, not the directivity). At $r = 2D^2/\lambda$ the path-length difference between the centre and the edge of the antenna is $\lambda/16$ (phase error $\pi/8$). For an antenna large compared with the wavelength, the reactive near field extends to about $r = 0.62\sqrt{D^3/\lambda}$ and the radiating near field from there to $2D^2/\lambda$. For an electrically small antenna ($D \ll \lambda$) the binding condition is $r \gg \lambda$; the radiation and induction terms of a Hertzian dipole are equal in magnitude at $r = \lambda/(2\pi)$.

TIME-AVERAGE POWER DENSITY AND TOTAL RADIATED POWER

$$\mathbf{S}_{\mathrm{av}} = \tfrac{1}{2}\,\mathrm{Re}\,(\mathbf{E} \times \mathbf{H}^*) = \frac{|E_\theta|^2 + |E_\phi|^2}{2\eta}\,\hat{a}_r, \qquad P_{\mathrm{rad}} = \oint_S \mathbf{S}_{\mathrm{av}} \cdot d\mathbf{s}$$

The middle form holds in the far field only, where E and H are in phase and $|E| = \eta|H|$. E is a peak phasor. In a lossless medium P_{rad} is the same through every closed surface that encloses the antenna.

HERTZIAN DIPOLE: FAR FIELD

$$E_\theta = \frac{j\eta\beta I_0\, dl}{4\pi r}\sin\theta\, e^{-j\beta r} = \frac{j\eta I_0\, dl}{2\lambda r}\sin\theta\, e^{-j\beta r}, \qquad H_\phi = \frac{E_\theta}{\eta}$$

Valid for $\beta r \gg 1$. In free space $|E_\theta| = \dfrac{60\pi I_0\, dl}{\lambda r}\sin\theta$ V/m. The pattern is $\propto \sin\theta$: maximum broadside ($\theta = 90^\circ$), nulls along the axis ($\theta = 0^\circ, 180^\circ$), and the same for every ϕ (omnidirectional in the plane $\theta = 90^\circ$).

HERTZIAN DIPOLE: RADIATED POWER AND RADIATION RESISTANCE

$$P_{\text{rad}} = \frac{\pi\eta}{3}\left(\frac{I_0\,dl}{\lambda}\right)^2 = 40\pi^2\left(\frac{dl}{\lambda}\right)^2 I_0^2, \qquad R_{\text{rad}} = \frac{2P_{\text{rad}}}{I_0^2} = 80\pi^2\left(\frac{dl}{\lambda}\right)^2 \approx 790\left(\frac{dl}{\lambda}\right)^2\ \Omega$$

The numerical forms assume free space ($\eta = 120\pi$); I_0 is the peak current. R_{rad} grows as $(dl/\lambda)^2$, which is why electrically short antennas are poor radiators: a small R_{rad} competes with the ohmic loss resistance.

SHORT DIPOLE (CURRENT FALLING LINEARLY TO ZERO AT THE ENDS)

$$R_{\text{rad}} = 20\pi^2\left(\frac{l}{\lambda}\right)^2$$

Centre-fed wire of total length $l \ll \lambda$ (roughly $l \le \lambda/10$) with a triangular current: I_0 at the feed, zero at the ends. Its average current is $I_0/2$, so its far field is half that of a Hertzian dipole of the same length with current I_0: power and R_{rad} are one quarter. The pattern ($\sin\theta$), directivity (1.5) and beamwidth (90°) are unchanged.

RADIATION INTENSITY AND TOTAL RADIATED POWER

$$U(\theta,\phi) = r^2 S_{\text{av}} = \frac{r^2}{2\eta}\left(|E_\theta|^2 + |E_\phi|^2\right), \qquad P_{\text{rad}} = \int_0^{2\pi}\int_0^{\pi} U(\theta,\phi)\,\sin\theta\,d\theta\,d\phi$$

Units W/sr. The element of solid angle is $d\Omega = \sin\theta\,d\theta\,d\phi$; dropping the $\sin\theta$ is an easy error to make.

DIRECTIVITY

$$D(\theta,\phi) = \frac{4\pi U(\theta,\phi)}{P_{\text{rad}}}, \qquad D_0 = \frac{4\pi U_{\max}}{P_{\text{rad}}} = \frac{4\pi}{\Omega_A}, \qquad D_0(\text{dBi}) = 10\log_{10} D_0$$

A pure ratio set by the shape of the pattern alone; losses do not enter. 'Directivity' with no direction stated means the maximum, D_0. dBi means decibels relative to an isotropic radiator; values relative to a half-wave dipole (dBd) are 2.15 dB lower.

BEAM SOLID ANGLE

$$\Omega_A = \oint F_n(\theta,\phi)\,d\Omega, \qquad F_n(\theta,\phi) = \frac{U(\theta,\phi)}{U_{\max}}, \qquad \Omega_A \approx \Theta_1\Theta_2$$

F_n is the normalized power pattern ($= |E|^2/|E|^2_{\max}$). Ω_A is the solid angle through which all the power would flow if U stayed at $U_{\max}$ inside it and were zero outside. The approximation uses the half-power beamwidths Θ_1 and Θ_2 in the two principal planes, in radians, and holds only for a single narrow main lobe with small side lobes.

DIRECTIVITY FROM BEAMWIDTHS (PENCIL BEAMS ONLY)

$$D_0 \approx \frac{4\pi}{\Theta_1\Theta_2}\ \ (\Theta \text{ in rad}) \approx \frac{41253}{\Theta_1\Theta_2}\ \ (\Theta \text{ in degrees})$$

$41253 = 4\pi(180/\pi)^2$. It ignores the power in side lobes and so tends to overestimate D_0. Do not use it for broad patterns such as a dipole's; integrate instead.

RADIATION EFFICIENCY AND GAIN

$$e_r = \frac{P_{\text{rad}}}{P_{\text{in}}} = \frac{R_{\text{rad}}}{R_{\text{rad}} + R_{\text{loss}}}, \qquad G(\theta,\phi) = \frac{4\pi U(\theta,\phi)}{P_{\text{in}}} = e_r D(\theta,\phi), \qquad G_0 = e_r D_0$$

P_{in} = power accepted at the antenna terminals; R_{loss} is referred to the same current as R_{rad}. $G \le D$, with equality only for a lossless antenna. Gain as defined here excludes impedance mismatch. If the feed line of impedance Z_0 is mismatched to the antenna impedance Z_A, the realized gain is $(1-|\Gamma|^2)G$ with $\Gamma = (Z_A - Z_0)/(Z_A + Z_0)$.

EFFECTIVE APERTURE

$$A_e = \frac{\lambda^2}{4\pi} G, \qquad P_r = S_{\text{inc}}\, A_e$$

For a receiving antenna matched to its load and polarization-matched to an incident wave of power density S_{inc} arriving from the direction in which the gain is G. For a lossless antenna $A_e = \lambda^2 D/(4\pi)$, the maximum effective aperture A_{em}. Hertzian or short dipole: $3\lambda^2/(8\pi) \approx 0.119\lambda^2$; half-wave dipole: $\approx 0.13\lambda^2$. For a large aperture antenna (horn, reflector) of physical area A_p, $A_e \le A_p$, with equality only for uniform, in-phase illumination of a lossless aperture; the ratio A_e/A_p is the aperture efficiency.

CENTRE-FED DIPOLE OF ANY LENGTH: FAR FIELD

$$E_\theta = \frac{j\eta I_m}{2\pi r} \frac{\cos\left(\frac{\beta l}{2}\cos\theta\right) - \cos\left(\frac{\beta l}{2}\right)}{\sin\theta} e^{-j\beta r}, \qquad H_\phi = \frac{E_\theta}{\eta}$$

Thin wire with the sinusoidal current above, far field. $l = \lambda/2$ gives the half-wave result below. For $l = \lambda$ the pattern is $[1+\cos(\pi\cos\theta)]/\sin\theta$: $D_0 \approx 2.41$ (3.82 dBi), HPBW $\approx 48^\circ$, and $R_{\text{rad}} \approx 199\ \Omega$ referred to I_m. The broadside directivity peaks near $l = 1.25\lambda$ ($D_0 \approx 3.28$); longer dipoles break into more lobes and the broadside lobe shrinks.

HALF-WAVE DIPOLE: FAR FIELD

$$E_\theta = \frac{j\eta I_0}{2\pi r} \frac{\cos\left(\frac{\pi}{2}\cos\theta\right)}{\sin\theta} e^{-j\beta r} = \frac{j60 I_0}{r} \frac{\cos\left(\frac{\pi}{2}\cos\theta\right)}{\sin\theta} e^{-j\beta r}, \qquad H_\phi = \frac{E_\theta}{\eta}$$

Thin wire, free space for the second form, I_0 = peak current at the feed. Nulls along the wire axis, maximum at $\theta = 90^\circ$, omnidirectional in ϕ; the beam is slightly narrower than the Hertzian dipole's $\sin\theta$.

HALF-WAVE DIPOLE: RADIATION PARAMETERS

$$R_{\text{rad}} \approx 73\ \Omega, \quad Z_{\text{in}} \approx 73 + j42.5\ \Omega, \quad D_0 \approx 1.64\ (2.15\ \text{dBi}), \quad \text{HPBW} \approx 78^\circ, \quad A_{em} \approx 0.13\lambda^2$$

In free space $R_{\text{rad}} = \frac{\eta}{4\pi}\,\text{Cin}(2\pi) = 30 \times 2.4377 \approx 73.1\ \Omega$, where $\text{Cin}(x) = \int_0^x \frac{1-\cos t}{t}\,dt$, and $D_0 = \frac{\eta}{\pi R_{\text{rad}}} = \frac{120}{R_{\text{rad}}} \approx 1.64$. $P_{\text{rad}} = \frac{1}{2} I_0^2 R_{\text{rad}} \approx 36.5\, I_0^2$ W. The feed sits at a current maximum, so R_{rad} is also the input resistance. The half-power points are at $\theta \approx 51^\circ$ and 129°. The $+j42.5\ \Omega$ reactance is tuned out by making the dipole slightly shorter than $\lambda/2$ (typically about 0.47λ to 0.49λ, shorter for thicker wire); the resistance at resonance is then slightly below 73 Ω.

MONOPOLE FROM ITS EQUIVALENT DIPOLE

$$R_{\text{rad,mono}}(h) = \tfrac{1}{2} R_{\text{rad,dip}}(2h), \qquad Z_{\text{in,mono}}(h) = \tfrac{1}{2} Z_{\text{in,dip}}(2h), \qquad D_{0,\text{mono}}(h) = 2\, D_{0,\text{dip}}(2h)$$

Monopole of height h over an infinite, perfectly conducting ground plane, compared with a dipole of total length $2h$ with the same current distribution. Same $U_{\max}$, half the power, hence twice the directivity. A finite or lossy ground changes these values.

QUARTER-WAVE MONOPOLE

$$R_{\text{rad}} \approx 36.5\ \Omega, \qquad Z_{\text{in}} \approx 36.5 + j21.25\ \Omega, \qquad D_0 \approx 3.28\ (5.16\ \text{dBi}), \qquad A_{em} \approx 0.26\lambda^2$$

$h = \lambda/4$ over an infinite perfect ground plane: half the half-wave dipole's impedance, twice its directivity. The pattern above the ground is the half-wave dipole's, with its maximum along the ground plane ($\theta = 90^\circ$) and half power at $\theta \approx 51^\circ$, i.e. about 39° above the ground plane.

FRIIS TRANSMISSION EQUATION

$$\frac{P_r}{P_t} = G_t\, G_r \left(\frac{\lambda}{4\pi R}\right)^2 = \frac{A_{et}\, A_{er}}{\lambda^2 R^2}$$

Conditions: each antenna in the other's far field, free space with no obstacles or multipath, both antennas matched to their lines and to each other in polarization, and G_t, G_r taken in the direction of the other antenna. Use gains, not directivities, when the antennas have loss. $(4\pi R/\lambda)^2$ is the free-space path loss.

FRIIS EQUATION IN DECIBELS

$$P_r(\text{dBm}) = P_t(\text{dBm}) + G_t(\text{dBi}) + G_r(\text{dBi}) - L_{\text{FS}}, \qquad L_{\text{FS}} = 20\log_{10}\frac{4\pi R}{\lambda} \approx 32.44 + 20\log_{10} f_{\text{MHz}} + 20\log_{10} R_{\text{km}}\ \text{dB}$$

$P(\text{dBm}) = 10\log_{10}(P/1\ \text{mW})$. The constant 32.44 dB uses $c = 3\times10^8$ m/s (the exact speed of light gives 32.45 dB). With fixed gains, L_{FS} rises by 6 dB for every doubling of distance or of frequency. With fixed effective apertures instead, the received power rises with frequency, because λ^2 then sits in the denominator.

ARRAY FACTOR OF A UNIFORM LINEAR ARRAY

$$AF = \sum_{n=0}^{N-1} e^{jn\psi}, \qquad \psi = \beta d\cos\theta + \alpha, \qquad |AF| = \left|\frac{\sin(N\psi/2)}{\sin(\psi/2)}\right|, \qquad |AF|_n = \left|\frac{\sin(N\psi/2)}{N\sin(\psi/2)}\right|$$

N identical, identically oriented elements on the z-axis at spacing d, equal amplitudes, element n fed with $I_0 e^{jn\alpha}$ (α = progressive phase, positive when each element leads the one before it). θ is measured from the array axis; for an array along another axis use the cosine of the angle from that axis (for the x-axis, $\sin\theta\cos\phi$). $|AF|$ has its maximum value N at $\psi = 0$, and again at $\psi = \pm2\pi, \pm4\pi, \ldots$ if those values are reached.

TWO-ELEMENT ARRAY

$$|AF| = \left|1 + e^{j\psi}\right| = 2\left|\cos\frac{\psi}{2}\right|, \qquad \psi = \beta d\cos\theta + \alpha$$

Two equal-amplitude elements, the one at $z = d$ leading the one at $z = 0$ by α. Nulls where $\psi = \pm\pi$, provided that value lies in the visible range $\alpha - \beta d \le \psi \le \alpha + \beta d$ (for example, $d = \lambda/4$ with $\alpha = 0$ has no null at all). With unequal amplitudes, $I_2/I_1 = k$, $|AF| = |1 + k\,e^{j\psi}|$, whose minimum (reached where $\psi = \pm\pi$ is visible) is $|1 - k|$: there is no complete null unless $k = 1$.

MAIN-BEAM DIRECTION AND PHASING

$$\cos\theta_0 = -\frac{\alpha}{\beta d}; \qquad \text{broadside } (\theta_0 = 90^\circ):\ \alpha = 0; \qquad \text{end-fire toward } \theta_0 = 0^\circ:\ \alpha = -\beta d; \qquad \text{toward } \theta_0 = 180^\circ:\ \alpha = +\beta d$$

Set $\psi = 0$ in the wanted direction; a real beam direction exists only if $|\alpha| \le \beta d$. With the $e^{j\omega t}$ convention the beam leans toward the end whose elements lag: for end-fire along $+z$, each element lags the one before it by βd, exactly the phase a wave travelling along $+z$ picks up between elements. Varying α electronically steers the beam (a phased array).

NULLS AND SIDE LOBES OF A UNIFORM ARRAY

$$\psi_{\text{null}} = \pm\frac{2m\pi}{N}, \quad m = 1, 2, 3, \ldots\ (m \ne N, 2N, \ldots); \qquad \text{first side lobe} \approx -13.3\ \text{dB near } \psi \approx \pm\frac{3\pi}{N}\ \ (\text{large } N)$$

Solve $\beta d\cos\theta + \alpha = \psi_{\text{null}}$ for θ and keep only solutions with $|\cos\theta| \le 1$. Small arrays have higher side lobes (about -11.3 dB for $N = 4$). With uniform amplitudes, changing d or α only moves the side lobes in angle (or out of the visible range); lowering their level needs a tapered amplitude distribution.

BROADSIDE ARRAY: NULLS AND BEAMWIDTHS

$$\theta_{\text{null}} = \cos^{-1}\left(\pm\frac{m\lambda}{Nd}\right), \qquad \text{FNBW} = 2\left[\frac{\pi}{2} - \cos^{-1}\frac{\lambda}{Nd}\right] \approx \frac{2\lambda}{Nd}, \qquad \text{HPBW} \approx 0.886\,\frac{\lambda}{Nd}\ \text{rad} \approx 50.8^\circ\,\frac{\lambda}{Nd}$$

Uniform array with $\alpha = 0$; $m = 1, 2, \ldots$ with $m\lambda/(Nd) \le 1$ and m not a multiple of N. The approximate forms (in radians unless marked) need $Nd \gg \lambda$. The beamwidth depends on the array length Nd, not on N alone.

ORDINARY END-FIRE ARRAY: NULLS AND BEAMWIDTHS

$$\theta_{\text{null}} = \cos^{-1}\left(1 - \frac{m\lambda}{Nd}\right), \qquad \text{FNBW} = 2\cos^{-1}\left(1 - \frac{\lambda}{Nd}\right) \approx 2\sqrt{\frac{2\lambda}{Nd}}, \qquad \text{HPBW} \approx 2\cos^{-1}\left(1 - 0.443\,\frac{\lambda}{Nd}\right) \approx 2\sqrt{0.886\,\frac{\lambda}{Nd}}$$

Uniform array with $\alpha = -\beta d$, beam toward $\theta = 0^\circ$; $m = 1, 2, \ldots$ with $m\lambda/(Nd) \le 2$ and m not a multiple of N. The square-root approximations (radians) need $Nd \gg \lambda$. For the same length, the end-fire beam is much wider than the broadside beam in the plane containing the axis, but it is a pencil beam around the axis instead of a fan all round it.

CONDITION FOR NO GRATING LOBES

$$d < \lambda \ \text{(broadside)}, \qquad d < \frac{\lambda}{2} \ \text{(ordinary end-fire)}, \qquad d < \frac{\lambda}{1 + |\cos\theta_0|} \ \text{(beam steered to } \theta_0\text{)}$$

A grating lobe is a second full maximum ($|AF| = N$), where $\psi = \pm 2\pi$. As θ runs from 0° to 180°, ψ covers the visible range $\alpha - \beta d \le \psi \le \alpha + \beta d$; there is no grating lobe if this range contains no multiple of 2π other than the main beam's $\psi = 0$. At exactly $d = \lambda$ (broadside) grating lobes appear along the axis; at exactly $d = \lambda/2$ (end-fire) a second beam appears at $\theta = 180^\circ$.

DIRECTIVITY OF UNIFORM ARRAYS OF ISOTROPIC ELEMENTS (APPROXIMATE)

$$D_0 \approx 2N\frac{d}{\lambda} \ \text{(broadside)}, \qquad D_0 \approx 4N\frac{d}{\lambda} \ \text{(ordinary end-fire)}$$

Valid for $Nd \gg \lambda$ and no grating lobes. For a broadside array with $d = \lambda/2$, $D_0 = N$ exactly, for any N (see the exact formula below).

PATTERN MULTIPLICATION

$$E_{\text{total}}(\theta, \phi) = E_{\text{element}}(\theta, \phi) \times AF(\theta, \phi)$$

For identical, identically oriented elements; the array factor carries the feed amplitudes and phases, and mutual coupling is ignored. In dB the two patterns add. A null of either factor is a null of the total, and the total maximum lies where the product peaks, which need not be at the maximum of either factor.

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