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Control Systems Formula Sheet

INSIDE

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- When to use each one
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CLOSED-LOOP TRANSFER FUNCTION (NEGATIVE FEEDBACK)

$$T(s) = \frac{C(s)}{R(s)} = \frac{G(s)}{1 + G(s)H(s)}$$

Here the feedback signal is subtracted at the comparator. The characteristic equation is $1 + G(s)H(s) = 0$. After clearing fractions, its roots are the closed-loop poles.

CLOSED-LOOP TRANSFER FUNCTION (POSITIVE FEEDBACK)

$$T(s) = \frac{G(s)}{1 - G(s)H(s)}$$

Here the feedback signal is added at the comparator. The characteristic equation is $1 - G(s)H(s) = 0$.

ACTUATING-SIGNAL TRANSFER FUNCTION

$$\frac{E(s)}{R(s)} = \frac{1}{1 + G(s)H(s)}$$

This uses $E = R - HC$ with negative feedback. For unity feedback ($H = 1$), E is the true error $R - C$. With non-unity H, the actuating signal is not the reference-minus-output error.

DEFINITION OF THE TRANSFER FUNCTION

$$G(s) = \left.\frac{Y(s)}{U(s)}\right|_{\text{zero initial conditions}} = \mathcal{L}\{g(t)\}$$

Applies to LTI systems only. $g(t)$ is the unit-impulse response.

TRANSFER FUNCTION FROM A LINEAR DIFFERENTIAL EQUATION

$$a_n y^{(n)} + \cdots + a_1 \dot{y} + a_0 y = b_m u^{(m)} + \cdots + b_1 \dot{u} + b_0 u$$

$$G(s) = \frac{Y(s)}{U(s)} = \frac{b_m s^m + \cdots + b_1 s + b_0}{a_n s^n + \cdots + a_1 s + a_0}$$

With all initial conditions zero, each d^k/dt^k becomes s^k. The denominator polynomial set to zero is the characteristic equation.

POLE-ZERO (ROOT-LOCUS) FORM

$$G(s) = K\,\frac{(s - z_1)(s - z_2)\cdots(s - z_m)}{(s - p_1)(s - p_2)\cdots(s - p_n)}$$

The zeros z_i are the values of s that make $G(s) = 0$. The poles p_j are the roots of the denominator, where $G(s) \to \infty$. n is the order of the system. G is proper if $m \le n$ and strictly proper if $m < n$; physical systems are proper.

TIME-CONSTANT (BODE) FORM

$$G(s) = \frac{K_{dc}(1 + sT_1)(1 + sT_2)\cdots}{s^N(1 + s\tau_1)(1 + s\tau_2)\cdots}$$

N is the number of poles at the origin. For the loop $G(s)H(s)$, N is the **type**. With $N = 0$, $K_{dc} = G(0)$ is the DC gain, which is the steady-state output per unit step if the system is stable.

MASS-SPRING-DAMPER

$$\frac{X(s)}{F(s)} = \frac{1}{Ms^2 + Bs + K}$$

Derived from $M\frac{d^2x}{dt^2} + B\frac{dx}{dt} + Kx = f$. The natural frequency is $\omega_n = \sqrt{K/M}$, the damping ratio is $\zeta = \frac{B}{2\sqrt{KM}}$, and the DC gain (static compliance) is $1/K$.

SERIES RLC, OUTPUT ACROSS C

$$\frac{V_C(s)}{V_i(s)} = \frac{1}{LCs^2 + RCs + 1}$$

Series R, L and C driven by v_i. Here $\omega_n = 1/\sqrt{LC}$ and $\zeta = \frac{R}{2}\sqrt{C/L}$.

INVERTING OP-AMP STAGE

$$\frac{V_o(s)}{V_i(s)} = -\frac{Z_f(s)}{Z_i(s)}$$

This assumes an ideal op-amp (infinite open-loop gain, zero input current) with its non-inverting input grounded. Z_i is in series with the input and Z_f connects the output to the inverting input. With $Z_i = R_1$ and $Z_f = 1/(sC)$ the stage is an integrator, $-1/(sR_1C)$. A second inverting stage restores the sign. An ideal op-amp has zero output impedance, so cascaded stages do not load each other.

ARMATURE-CONTROLLED DC MOTOR

$$\frac{\Theta(s)}{V_a(s)} = \frac{K_t}{s\left[(R_a + sL_a)(Js + B) + K_tK_b\right]}$$

Motor torque is $T_m = K_t i_a$ and back emf is $e_b = K_b\, d\theta/dt$, with J and B referred to the motor shaft. The back emf is a built-in negative velocity feedback. In SI units, $K_t = K_b$ numerically. For speed output, drop the factor s: $\Omega/V_a = K_t/[(R_a + sL_a)(Js + B) + K_tK_b]$. With a load torque T_L opposing the motor, superposition gives $\Omega(s) = \dfrac{K_tV_a(s) - (R_a + sL_a)T_L(s)}{(R_a + sL_a)(Js + B) + K_tK_b}$.

DC MOTOR SPEED WITH NEGLIGIBLE ARMATURE INDUCTANCE

$$\frac{\Omega(s)}{V_a(s)} = \frac{K_m}{\tau_m s + 1}, \quad K_m = \frac{K_t}{R_aB + K_tK_b}, \quad \tau_m = \frac{R_aJ}{R_aB + K_tK_b}$$

Valid when $L_a \approx 0$. K_m is the steady-state speed per volt. The inertia J affects only the time constant, not the final speed.

GEAR TRAIN AND REFLECTED LOAD

$$\frac{\theta_2}{\theta_1} = \frac{N_1}{N_2}, \quad J_{eq} = J_1 + \left(\frac{N_1}{N_2}\right)^2 J_2, \quad B_{eq} = B_1 + \left(\frac{N_1}{N_2}\right)^2 B_2$$

Gear 1 (motor side) has N_1 teeth and gear 2 (load side) has N_2. To refer a load-side J_2, B_2 or stiffness to the motor side, multiply by $(N_1/N_2)^2$. For ideal lossless gears the torque scales as $T_2 = (N_2/N_1)T_1$, so a load torque T_L referred to the motor side becomes $(N_1/N_2)T_L$.

OUTPUT WITH REFERENCE, DISTURBANCE AND SENSOR NOISE

$$C(s) = \frac{G_1G_2}{1 + G_1G_2H}\left[R(s) - N(s)\right] + \frac{G_2}{1 + G_1G_2H}\,D(s)$$

Negative feedback. The noise term has the same transfer function as the reference, with a minus sign, so no amount of loop gain can separate the two.

SENSITIVITY (DEFINITION)

$$S_G^T = \frac{\partial T/T}{\partial G/G} = \frac{G}{T}\frac{\partial T}{\partial G}$$

The fractional change in T per fractional change in the parameter G, equal to $\partial \ln T/\partial \ln G$. It is valid for small changes. For a large change, compute T before and after instead.

CLOSED-LOOP SENSITIVITY TO G AND TO H

$$S_G^T = \frac{1}{1+GH}, \qquad S_H^T = \frac{-GH}{1+GH}$$

For negative feedback, $T = G/(1+GH)$. An open loop ($T = G$) has $S_G^T = 1$. When $|GH| \gg 1$, $S_G^T \to 0$ but $S_H^T \to -1$, so a feedback-path error reaches the output almost one-for-one. For the frequency-dependent sensitivity, evaluate at $s = j\omega$.

DISTURBANCE TRANSFER FUNCTION

$$\frac{C(s)}{D(s)} = \frac{G_2}{1+G_1G_2H} \approx \frac{1}{G_1H} \quad (|G_1G_2H| \gg 1)$$

For a disturbance entering after G_1. Compared with open loop, where $C/D = G_2$, feedback reduces the disturbance's effect by the factor $|1+G_1G_2H|$. In the high-gain limit, only the gain **upstream** of the disturbance point (and H) sets the residual effect $1/(G_1H)$; raising G_2 does not reduce it. For a disturbance added directly at the output, $C/D = 1/(1+GH)$.

GAIN-BANDWIDTH TRADE (FIRST-ORDER PLANT)

$$T(s) = \frac{K}{1+KH} \cdot \frac{1}{1+s\,\dfrac{\tau}{1+KH}}$$

Plant $K/(1+s\tau)$ with a constant negative-feedback gain H (use $H = 1$ for unity feedback). The DC gain falls by the factor $(1+KH)$. The bandwidth rises from $1/\tau$ to $(1+KH)/\tau$ rad/s, and the time constant shrinks by the same factor. The gain-bandwidth product stays K/τ.

EQUIVALENT UNITY-FEEDBACK FORWARD PATH

$$G_{eq}(s) = \frac{G(s)}{1+G(s)H(s)-G(s)}$$

A unity negative-feedback loop around G_{eq} has the same closed-loop transfer function $G/(1+GH)$. Use it when a steady-state-error formula assumes unity feedback, so that $E = R - C$ is the true error.

MASON'S GAIN FORMULA

$$T = \frac{\sum_k P_k\,\Delta_k}{\Delta}$$

The gain from a source node to any other node. P_k is the gain of the k-th forward path. Δ is the graph determinant. Δ_k is Δ recomputed with every loop that touches the k-th forward path removed.

GRAPH DETERMINANT

$$\Delta = 1 - \sum_i L_i + \sum_{i,j} L_iL_j - \sum_{i,j,k} L_iL_jL_k + \cdots$$

The second sum runs over every pair of mutually non-touching loops, the third over every triple, and so on. Loop gains carry their own signs, so a negative-feedback loop ($L = -GH$) adds $+GH$ to Δ. $\Delta = 0$ is the characteristic equation, and Δ is the same for every input-output pair in a given graph.

Time response and steady-state error

FIRST-ORDER UNIT-STEP RESPONSE

$$c(t) = K\left(1 - e^{-t/\tau}\right), \quad t \geq 0$$

For a unit step into $T(s) = K/(\tau s+1)$. At $t = \tau$ the output has made 63.2% of its final change, and its initial slope is K/τ.

FIRST-ORDER SPECIFICATIONS

$$t_d\ (50\%) = \tau \ln 2 \approx 0.69\,\tau$$
$$t_r\ (10 \text{ to } 90\%) = \tau \ln 9 \approx 2.2\,\tau$$
$$t_s\ (2\%) \approx 4\,\tau$$
$$t_s\ (5\%) \approx 3\,\tau$$

The exact settling values are $-\ln 0.02 = 3.91$ and $-\ln 0.05 = 3.00$ time constants. A first-order system without a zero has no peak time and no overshoot.

FIRST-ORDER UNIT-RAMP RESPONSE

$$c(t) = t - \tau + \tau e^{-t/\tau}, \qquad e_{ss} = \tau$$

For $T(s) = 1/(\tau s + 1)$, which has unity DC gain. After the transient, the output runs parallel to the input but τ seconds behind it, so the steady-state error to a unit ramp is τ. This is the same as a unity-feedback loop with $G(s) = 1/(\tau s)$, whose $K_v = 1/\tau$.

STANDARD SECOND-ORDER FORM

$$T(s) = \frac{\omega_n^2}{s^2 + 2\zeta\omega_n s + \omega_n^2}$$

Unity DC gain. If the numerator is some other constant, ζ and ω_n still come from the denominator. Only the final value changes.

UNDERDAMPED POLE LOCATION

$$s_{1,2} = -\zeta\omega_n \pm j\omega_n\sqrt{1-\zeta^2} = -\sigma \pm j\omega_d, \quad |s_{1,2}| = \omega_n, \quad \cos\theta = \zeta$$

Holds for $0 < \zeta < 1$. Here $\sigma = \zeta\omega_n$ is the decay rate, $\omega_d = \omega_n\sqrt{1-\zeta^2}$ is the damped frequency, and θ is the angle of the pole measured from the **negative real axis**.

UNDERDAMPED UNIT-STEP RESPONSE

$$c(t) = 1 - \frac{e^{-\zeta\omega_n t}}{\sqrt{1-\zeta^2}}\sin\left(\omega_d t + \theta\right), \quad \theta = \cos^{-1}\zeta$$

For $0 < \zeta < 1$ in the standard form. The oscillation stays inside the envelope $1 \pm e^{-\zeta\omega_n t}/\sqrt{1-\zeta^2}$. The settling-time formula is based on this envelope.

UNITY FEEDBACK AROUND A TYPE-1 SECOND-ORDER PLANT

$$G(s) = \frac{K}{s(s+a)} \;\Rightarrow\; T(s) = \frac{K}{s^2 + as + K}, \quad \omega_n = \sqrt{K}, \quad \zeta = \frac{a}{2\sqrt{K}}$$

Unity negative feedback, with $K > 0$ and $a > 0$. The loop is critically damped at $K = a^2/4$ and underdamped for $K > a^2/4$. Raising K raises ω_n and lowers ζ. The real part of the poles, $\sigma = \zeta\omega_n = a/2$, stays fixed as long as $\zeta < 1$.

RISE TIME (0 TO 100%)

$$t_r = \frac{\pi - \theta}{\omega_d}, \quad \theta = \cos^{-1}\zeta \text{ (radians)}$$

Underdamped systems only. θ must be in radians.

PEAK TIME

$$t_p = \frac{\pi}{\omega_d} = \frac{\pi}{\omega_n\sqrt{1-\zeta^2}}$$

Half of the damped period $T_d = 2\pi/\omega_d$. It depends only on the imaginary part of the poles.

PEAK OVERSHOOT

$$M_p = e^{-\zeta\pi/\sqrt{1-\zeta^2}} \quad (\times 100 \text{ for percent})$$

Valid for $0 < \zeta < 1$, and depends only on ζ. For $\zeta \geq 1$ the step response has no overshoot.

DAMPING RATIO FROM OVERSHOOT

$$\zeta = \frac{-\ln M_p}{\sqrt{\pi^2 + (\ln M_p)^2}}$$

Enter M_p as a fraction (0.163, not 16.3). This is what you use in design questions that specify the overshoot.

SETTLING TIME

$$t_s \approx \frac{4}{\zeta\omega_n}\ (2\%\ \text{band}), \qquad t_s \approx \frac{3}{\zeta\omega_n}\ (5\%\ \text{band})$$

Envelope approximations for $0 < \zeta < 1$. They give the time it takes $e^{-\zeta\omega_n t}$ to fall to about 2% ($e^{-4} = 1.8\%$) or 5% ($e^{-3} = 5.0\%$), and $1/(\zeta\omega_n)$ is the time constant of the envelope. They are not the exact time at which the response enters the band for good. For the worked example below, the 2% formula gives 1.33 s and the exact value is about 1.19 s. Use these formulas unless the question gives a different rule.

FINAL VALUE THEOREM

$$\lim_{t\to\infty} f(t) = \lim_{s\to 0} sF(s)$$

Valid only if every pole of $sF(s)$ lies in the open left half plane, that is, only if $f(t)$ actually settles. It gives wrong answers for unstable systems and for sinusoidal inputs (which put poles on the imaginary axis).

STEADY-STATE OUTPUT FOR A STEP

$$c(\infty) = \lim_{s\to 0} s\,T(s)\,\frac{A}{s} = A\,T(0)$$

For a step of size A into a stable closed loop $T(s) = C(s)/R(s)$. So for any loop structure, the step error $r - c$ settles at $A\,[1 - T(0)]$.

STEADY-STATE ERROR

$$e_{ss} = \lim_{s\to 0} \frac{sR(s)}{1 + G(s)}\ \ (\text{unity feedback}), \qquad e_{ss} = \lim_{s\to 0} sR(s)\,[1 - T(s)]\ \ (\text{any loop})$$

The error here is $r - c$. The first form needs unity feedback. The second works for any loop structure, including non-unity feedback, as long as you know the closed-loop $T(s) = C(s)/R(s)$. Both need a stable closed loop. For a ramp or parabolic input the limit can come out infinite, which means the error grows without bound.

SYSTEM TYPE

$$G(s)H(s) = \frac{K\,(1 + sT_{z1})(1 + sT_{z2})\cdots}{s^N(1 + sT_{p1})(1 + sT_{p2})\cdots}, \qquad \text{type} = N$$

N is the number of open-loop poles at the origin (integrators), counted after cancelling any factor of s common to the numerator and denominator. K here is the gain in time-constant form. Type belongs to the open-loop function, not to $T(s)$, and it has nothing to do with the order of the system. With unity feedback, the type of $G(s)$ decides which standard inputs give zero, finite or unbounded $r - c$ error. With non-unity feedback, the type of $G(s)H(s)$ describes the actuating error $R - HC$, not $r - c$.

STATIC ERROR CONSTANTS

$$K_p = \lim_{s\to 0} G(s)H(s), \qquad K_v = \lim_{s\to 0} s\,G(s)H(s), \qquad K_a = \lim_{s\to 0} s^2 G(s)H(s)$$

The position, velocity and acceleration constants. For unity feedback, $H(s) = 1$ and these constants give the $r - c$ error. With non-unity feedback, constants taken from $G(s)H(s)$ give the actuating error $R - HC$.

STEADY-STATE ERRORS FOR STANDARD INPUTS

$$e_{ss}\big|_{A\,u(t)} = \frac{A}{1 + K_p}, \qquad e_{ss}\big|_{At\,u(t)} = \frac{A}{K_v}, \qquad e_{ss}\big|_{\frac{At^2}{2}u(t)} = \frac{A}{K_a}$$

For unity feedback and a stable closed loop. When the input is a sum of these inputs, the errors add (linearity).

STEADY-STATE ERROR DUE TO A DISTURBANCE

$$C(s) = \frac{G_c G_p}{1 + G_c G_p} R(s) + \frac{G_p}{1 + G_c G_p} D(s), \qquad e_{ss,D} = -\lim_{s \to 0} \frac{s\, G_p(s)\, D(s)}{1 + G_c(s) G_p(s)}$$

Unity negative feedback, with controller $G_c(s)$ followed by plant $G_p(s)$, and disturbance $D(s)$ added at the plant input (between G_c and G_p). With $e = r - c$, $e_{ss,D}$ is the part of the steady-state error caused by the disturbance. Find it with $r = 0$, then add it to the error caused by r. The closed loop must be stable. For a step disturbance of size D_0: if G_c contains an integrator, $e_{ss,D} = 0$. If only G_p has an integrator and $G_c(0)$ is finite and nonzero, $e_{ss,D} = -D_0 / G_c(0)$. If neither has one, $e_{ss,D} = -D_0\, G_p(0) / \left[1 + G_c(0) G_p(0)\right]$.

Stability and the Routh–Hurwitz criterion

BIBO STABILITY CONDITION

$$\int_0^\infty |h(t)|\, dt < \infty \quad \Longleftrightarrow \quad \mathrm{Re}(p_i) < 0 \ \text{ for every pole } p_i \text{ of } T(s)$$

For a causal LTI system with rational, proper $T(s)$ (numerator degree $\leq$ denominator degree), with poles counted after cancelling common factors. If $T(s)$ is proper but not strictly proper, $h(t)$ also contains an impulse $D\,\delta(t)$, which does not affect BIBO stability. An improper $T(s)$, such as the pure differentiator $T(s) = s$, is never BIBO stable. A single pole on the $j\omega$-axis, even a simple one, is enough to make the system not BIBO stable.

CLOSED-LOOP CHARACTERISTIC EQUATION (NEGATIVE FEEDBACK)

$$T(s) = \frac{G(s)}{1 + G(s)H(s)}, \qquad 1 + G(s)H(s) = 0 \ \Rightarrow \ D_G(s) D_H(s) + N_G(s) N_H(s) = 0$$

Here $G = N_G / D_G$ and $H = N_H / D_H$, written without cancelling any factor. The roots are the closed-loop poles. For positive feedback, $T(s) = \frac{G(s)}{1 - G(s)H(s)}$ and the characteristic equation is $1 - G(s)H(s) = 0$, that is $D_G D_H - N_G N_H = 0$.

NECESSARY CONDITION FOR STABILITY

$$a_0, a_1, a_2, \ldots, a_n \ \text{ all present (non-zero) and of the same sign}$$

Necessary for every order n. It is necessary **and sufficient** only for $n = 1$ and $n = 2$. A missing term or a sign change proves that not every root is in the LHP, but it does not tell you how many roots are in the RHP or whether some lie on the $j\omega$-axis.

ROUTH ARRAY ENTRIES

$$b_1 = \frac{a_1 a_2 - a_0 a_3}{a_1}, \quad b_2 = \frac{a_1 a_4 - a_0 a_5}{a_1}, \quad c_1 = \frac{b_1 a_3 - a_1 b_2}{b_1}, \quad c_2 = \frac{b_1 a_5 - a_1 b_3}{b_1}$$

General rule: with rows $i - 2$ and $i - 1$ directly above row i, the entry in column j is $x_{i,j} = \frac{x_{i-1,1}\, x_{i-2,j+1} - x_{i-2,1}\, x_{i-1,j+1}}{x_{i-1,1}}$. The divisor is always the first-column entry of the row directly above. Take any missing entry at the right-hand end as 0. In an array with no special case, the single s^0 entry equals a_n (or a positive multiple of it if you rescaled a row), which is a quick arithmetic check.

ROUTH'S CRITERION

$$\text{number of roots with } \mathrm{Re}(s) > 0 \ = \ \text{number of sign changes in the first column}$$

Applies when no first-column entry is zero. With $a_0 > 0$, the system is stable if and only if every first-column entry is positive. If the array is completed with no zero in the first column, no root lies on the $j\omega$-axis, so all the remaining roots are in the LHP.

THIRD-ORDER SHORTCUT

$$a_0 s^3 + a_1 s^2 + a_2 s + a_3 = 0: \quad \text{stable} \iff a_0, a_1, a_2, a_3 > 0 \ \text{ and } \ a_1 a_2 > a_0 a_3$$

Assumes $a_0 > 0$. This comes from the s^1 entry $(a_1 a_2 - a_0 a_3)/a_1$. If $a_1 a_2 = a_0 a_3$ with every coefficient positive, the polynomial factors as $(a_0 s^2 + a_2)\left(s + \frac{a_1}{a_0}\right)$. That gives a pair of roots at $s = \pm j\sqrt{a_2/a_0}$ (equivalently $\pm j\sqrt{a_3/a_1}$) and a third root at $-a_1/a_0$, which is marginal stability.

FOURTH-ORDER SHORTCUT

$$a_0 s^4 + a_1 s^3 + a_2 s^2 + a_3 s + a_4 = 0: \quad \text{stable} \iff \text{all } a_i > 0 \ \text{ and } \ a_1 a_2 a_3 > a_0 a_3^2 + a_1^2 a_4$$

Assumes $a_0 > 0$. This comes from requiring the s^2 entry $(a_1 a_2 - a_0 a_3)/a_1$ and the s^1 entry to be positive. When every coefficient is positive, this condition already implies $a_1 a_2 > a_0 a_3$.

HURWITZ DETERMINANT FORM (EQUIVALENT TO ROUTH)

$$H = \begin{bmatrix} a_1 & a_3 & a_5 & \cdots \\ a_0 & a_2 & a_4 & \cdots \\ 0 & a_1 & a_3 & \cdots \\ 0 & a_0 & a_2 & \cdots \\ \vdots & \vdots & \vdots & \ddots \end{bmatrix}_{n \times n}, \qquad \text{stable} \iff \Delta_1 > 0,\ \Delta_2 > 0,\ \ldots,\ \Delta_n > 0$$

Assumes $a_0 > 0$. Row i, column j of H is a_{2j-i}, with $a_k = 0$ for $k < 0$ or $k > n$. Δ_i is the determinant of the top-left $i \times i$ block, so $\Delta_1 = a_1$, $\Delta_2 = a_1 a_2 - a_0 a_3$ and $\Delta_n = a_n \Delta_{n-1}$. When no Δ_i is zero, the Routh first column is $a_0,\ \Delta_1,\ \frac{\Delta_2}{\Delta_1},\ \frac{\Delta_3}{\Delta_2},\ \ldots,\ \frac{\Delta_n}{\Delta_{n-1}}$, so the two tests always agree.

AUXILIARY POLYNOMIAL

$$A(s) = c_1 s^k + c_2 s^{k-2} + c_3 s^{k-4} + \ldots, \qquad \text{zero row} \leftarrow \text{coefficients of } \frac{dA}{ds}$$

$c_1, c_2, \ldots$ are the entries of row s^k, directly above the zero row (a positive multiple of that row works equally well). $A(s)$ divides the characteristic polynomial exactly.

COUNTING THE ROOTS OF A(S)

$$N_{\text{RHP}}(A) = N_{\text{LHP}}(A) = p, \qquad N_{j\omega}(A) = k - 2p$$

k is the degree of $A(s)$ and p is the number of sign changes from the $A(s)$ row down to the s^0 row. If $p = 0$, all k roots of $A(s)$ lie on the $j\omega$-axis. If a second row of zeros appears lower down, $A(s)$ has repeated roots; see the last tip in this section.

MARGINAL GAIN AND FREQUENCY OF OSCILLATION

$$\text{At } K = K_{\text{mar}}: \quad A(s) = \alpha s^2 + \beta = 0 \ \Rightarrow \ s = \pm j\omega, \quad \omega = \sqrt{\frac{\beta}{\alpha}} \ \text{rad/s}$$

α and β are the s^2-row entries evaluated at the K that makes the s^1 row zero. Both must be positive for the roots to lie on the $j\omega$-axis. If the limit is set by the s^0 entry going to zero instead, the boundary root is at $s = 0$ and there is no oscillation.

SHORTCUT: THREE REAL OPEN-LOOP POLES

$$G(s) = \frac{K}{(s+a)(s+b)(s+c)}: \quad K_{\text{mar}} = (a+b)(b+c)(c+a), \qquad \omega = \sqrt{ab + bc + ca}\ \text{rad/s}$$

Unity negative feedback, real $a, b, c \geq 0$. Put $c = 0$ for $G(s) = \frac{K}{s(s+a)(s+b)}$, which gives $K_{\text{mar}} = ab(a+b)$ and $\omega = \sqrt{ab}$. The characteristic equation is $s^3 + (a+b+c)s^2 + (ab+bc+ca)s + abc + K = 0$. The third-order shortcut, with $(a+b+c)(ab+bc+ca) - abc = (a+b)(b+c)(c+a)$, gives stability for $-abc < K < (a+b)(b+c)(c+a)$. At K_{mar} the third root is $s = -(a+b+c)$. For any other form of $G(s)$, build the array.

RELATIVE STABILITY: SHIFTING THE AXIS

$$s = z - \sigma \;\; (\sigma > 0): \quad \text{sign changes in the } z\text{-array} = \text{number of roots with } \mathrm{Re}(s) > -\sigma$$

Substitute into the characteristic polynomial, expand in z and apply Routh. The count holds when no special case arises. No sign change and no zero in the first column means every root lies strictly to the left of the line $s = -\sigma$; a row of zeros means some roots lie on the line.

Root locus

CLOSED-LOOP CHARACTERISTIC EQUATION

$$1 + K\,G(s)H(s) = 0 \quad \Longleftrightarrow \quad D(s) + K\,N(s) = 0$$

Negative feedback, with $G(s)H(s) = N(s)/D(s)$ in monic form. The ordinary root locus is for $K \geq 0$.

ANGLE CONDITION

$$\angle G(s)H(s) = \sum_{j=1}^{m} \angle(s - z_j) - \sum_{i=1}^{n} \angle(s - p_i) = \pm(2q+1)\,180^\circ, \quad q = 0, 1, 2, \ldots$$

A point s that is not itself an open-loop pole or zero is on the root locus ($K \geq 0$) if and only if the net angle is an odd multiple of 180°. K does not appear, so this condition alone decides the shape of the locus.

MAGNITUDE CONDITION (GAIN AT A POINT)

$$K = \frac{1}{|G(s)H(s)|} = \frac{\prod_{i=1}^{n} |s - p_i|}{\prod_{j=1}^{m} |s - z_j|}$$

Use this only at a point that already satisfies the angle condition. It is the product of the distances to all poles divided by the product of the distances to all zeros. If $m = 0$ the denominator is 1.

ASYMPTOTE ANGLES

$$\theta_q = \frac{(2q+1)\,180^\circ}{n - m}, \quad q = 0, 1, \ldots, (n - m - 1)$$

For $K \geq 0$. Angles are measured counter-clockwise from the positive real axis. There are exactly $n - m$ asymptotes.

CENTROID (WHERE THE ASYMPTOTES MEET)

$$\sigma_A = \frac{\sum_{i=1}^{n} p_i - \sum_{j=1}^{m} z_j}{n - m}$$

Use the signed values: a pole at -3 contributes -3. The imaginary parts of conjugate pairs cancel, so σ_A is always real. You only need it when $n - m \geq 2$. When $n - m = 1$, the single asymptote is the negative real axis.

BREAKAWAY / BREAK-IN CONDITION

$$\frac{dK}{ds} = 0 \;\text{ with }\; K = -\frac{D(s)}{N(s)} \quad \Longleftrightarrow \quad N(s)\,D'(s) - D(s)\,N'(s) = 0$$

The roots are only candidates. Keep a root only if it lies on the locus: a real root must sit on a valid real-axis segment, and K must come out real and positive.

ANGLE OF DEPARTURE FROM A SIMPLE COMPLEX POLE

$$\phi_d = 180^\circ - \sum_{i \neq k} \angle(p_k - p_i) + \sum_{j=1}^{m} \angle(p_k - z_j)$$

For a pole p_k that is not repeated. Use the angles of the vectors drawn from every other pole and from every zero to p_k. Add or subtract 360° to bring the result into $(-180^\circ, 180^\circ]$. The conjugate pole departs at $-\phi_d$.

ANGLE OF ARRIVAL AT A SIMPLE COMPLEX ZERO

$$\phi_a = 180^\circ - \sum_{j \neq k} \angle(z_k - z_j) + \sum_{i=1}^{n} \angle(z_k - p_i)$$

For a zero z_k that is not repeated. Use the angles of the vectors drawn from every other zero and from every pole to z_k. The conjugate zero has arrival angle $-\phi_a$.

IMAGINARY-AXIS CROSSING FROM THE ROUTH ARRAY

$$\text{row of zeros at } K = K_{mar}; \qquad A(s) = a\,s^2 + b = 0 \;\Rightarrow\; \omega = \sqrt{\frac{b}{a}}$$

$A(s)$ is the auxiliary polynomial from the row just above the zero row (usually the s^2 row), evaluated at $K_{mar} > 0$. b usually depends on K, so substitute K_{mar} before taking the square root. This is a crossing only if $b/a > 0$ at K_{mar}. If $b/a < 0$, the zero row signals a pair of real roots $\pm\sqrt{-b/a}$ instead.

SUM OF CLOSED-LOOP POLES (POLE EXCESS AT LEAST 2)

$$\sum_{i=1}^{n} r_i(K) = \sum_{i=1}^{n} p_i \qquad (n - m \geq 2)$$

When $n - m \geq 2$, K does not appear in the s^{n-1} coefficient of $D(s) + KN(s)$, so the closed-loop poles r_i always add up to the sum of the open-loop poles. Use it to find the remaining real pole at the $j\omega$ crossing without dividing polynomials.

CONSTANT DAMPING-RATIO LINE

$$s = -\zeta\omega_n \pm j\omega_n\sqrt{1-\zeta^2}, \qquad \theta = \cos^{-1}\zeta$$

For $0 < \zeta < 1$. Closed-loop poles with damping ratio ζ lie on the two radial lines from the origin at angle θ above and below the negative real axis. The gain for a required ζ is K, from the magnitude condition, at the point where this line cuts the locus. If the line cuts the locus more than once, each crossing gives a valid gain.

EQUIVALENT LOOP FOR A VARYING PARAMETER

$$Q(s) + \alpha\,P(s) = 0 \quad \Longleftrightarrow \quad 1 + \alpha\,\frac{P(s)}{Q(s)} = 0$$

Valid when the characteristic polynomial is linear in the parameter $\alpha \geq 0$. Apply every rule to $P(s)/Q(s)$ with α in place of K. The poles of this equivalent loop are the roots of $Q(s)$, i.e. the closed-loop poles at $\alpha = 0$. They are generally **not** the open-loop poles of the physical system. If $P(s)$ or $Q(s)$ is not monic, move the leading coefficients into the gain first.

ANGLE AND MAGNITUDE CONDITIONS, COMPLEMENTARY LOCUS

$$\angle G(s)H(s) = \pm q\,360^\circ, \quad q = 0, 1, 2, \ldots \qquad |K| = \frac{\prod_{i=1}^{n} |s - p_i|}{\prod_{j=1}^{m} |s - z_j|}$$

Applies to $K \leq 0$ with negative feedback, or to $K \geq 0$ with positive feedback.

ASYMPTOTE ANGLES, COMPLEMENTARY LOCUS

$$\theta_q = \frac{q\,360^\circ}{n - m}, \quad q = 0, 1, \ldots, (n - m - 1)$$

The centroid formula does not change. When $n > m$ there is always an asymptote at 0°, i.e. along the positive real axis.

Frequency response: Bode and polar plots, gain and phase margins

SINUSOIDAL STEADY-STATE RESPONSE

$$u(t) = A\sin\omega t \;\Rightarrow\; y_{ss}(t) = A\,|G(j\omega)|\,\sin\left(\omega t + \angle G(j\omega)\right)$$

Valid only for a stable $G(s)$, after the transients decay. The frequency does not change, only the amplitude and phase.

TIME-CONSTANT FORM: MAGNITUDES ADD IN DB, PHASES ADD

$$G(s) = \frac{K\prod_i(1+sT_{zi})}{s^N\prod_k(1+sT_{pk})}$$

$$|G(j\omega)|_{\text{dB}} = 20\log_{10}K + \sum_i 20\log_{10}\sqrt{1+\omega^2T_{zi}^2} - 20N\log_{10}\omega - \sum_k 20\log_{10}\sqrt{1+\omega^2T_{pk}^2}$$

$$\angle G(j\omega) = \sum_i \tan^{-1}(\omega T_{zi}) - 90^\circ N - \sum_k \tan^{-1}(\omega T_{pk})$$

For $K>0$ and real poles and zeros in the left half plane ($T_{zi}, T_{pk} > 0$). A quadratic factor adds its own magnitude and phase terms (see the Bode table below). N is the system type, the number of poles at the origin. This additive structure is why Bode plots are built factor by factor.

VALUE AT THE NATURAL FREQUENCY

$$u = 1: \quad |T(j\omega_n)| = \frac{1}{2\zeta}, \qquad \angle T(j\omega_n) = -90^\circ$$

Holds for every $\zeta > 0$. In dB the value is $-20\log_{10}(2\zeta)$, which is 0 dB exactly when $\zeta = 0.5$.

RESONANT FREQUENCY

$$\omega_r = \omega_n\sqrt{1-2\zeta^2}$$

Only for $0 < \zeta < 1/\sqrt{2} \approx 0.707$. In that range $\omega_r < \omega_d < \omega_n$, where $\omega_d = \omega_n\sqrt{1-\zeta^2}$.

RESONANT PEAK

$$M_r = \frac{1}{2\zeta\sqrt{1-\zeta^2}}$$

Only for $0 < \zeta < 0.707$. For $\zeta \geq 0.707$ there is no peak, $|T|$ falls monotonically, and $M_r = 1$ at $\omega = 0$. M_r depends on ζ alone.

BANDWIDTH OF THE STANDARD SECOND-ORDER SYSTEM

$$\omega_B = \omega_n\sqrt{(1-2\zeta^2) + \sqrt{4\zeta^4 - 4\zeta^2 + 2}}$$

The frequency where $|T| = 1/\sqrt{2}$ (−3 dB relative to $|T(0)| = 1$). Valid for every $\zeta > 0$. $\omega_B = \omega_n$ exactly at $\zeta = 1/\sqrt{2}$ and $\omega_B = 0.644\,\omega_n$ at $\zeta = 1$.

BANDWIDTH OF A FIRST-ORDER SYSTEM

$$T(s) = \frac{1}{1+s\tau} \;\Rightarrow\; \omega_B = \frac{1}{\tau}$$

The bandwidth equals the corner frequency. A smaller time constant gives a wider bandwidth and a faster response.

PHASE MARGIN OF THE STANDARD SECOND-ORDER LOOP

$$\omega_{gc} = \omega_n\sqrt{\sqrt{1+4\zeta^4} - 2\zeta^2}, \qquad \text{PM} = \tan^{-1}\left(\frac{2\zeta}{\sqrt{\sqrt{1+4\zeta^4} - 2\zeta^2}}\right)$$

For open loop $G(s) = \omega_n^2/[s(s+2\zeta\omega_n)]$ with unity negative feedback. GM is infinite because the phase never reaches -180°. Rough approximation: PM in degrees $\approx 100\zeta$ for $0 < \zeta \leq 0.6$, off by at most about 3.5°. Use the exact formula when the options are close.

DECIBELS, DECADES AND SLOPES

$$M_{\text{dB}} = 20\log_{10}|G(j\omega)|, \qquad M(\omega_2) = M(\omega_1) + S\log_{10}\frac{\omega_2}{\omega_1}$$

S is the slope in dB/decade, and the relation holds along one straight segment. A decade is a factor of 10 and an octave a factor of 2, so 20 dB/decade ≈ 6.02 dB/octave.

LOW-FREQUENCY ASYMPTOTE AND ITS 0 DB CROSSING

$$|G|_{\text{dB, LF}} = 20\log_{10}K - 20N\log_{10}\omega, \qquad \omega_{0\,\text{dB}} = K^{1/N} \;\; (N \geq 1)$$

K must be the time-constant-form gain. The line (extended if necessary) reads $20\log_{10}K$ at $\omega = 1$. Type 0: the plot starts flat at $20\log_{10}K$.

STATIC ERROR CONSTANTS FROM THE LOW-FREQUENCY ASYMPTOTE

$$\text{Type 0: } K_p = K = 10^{M_{\text{LF}}/20}, \qquad \text{Type 1: } K_v = K = \omega_{0\,\text{dB}}, \qquad \text{Type 2: } K_a = K = \omega_{0\,\text{dB}}^2$$

For unity feedback, where $K_p = \lim_{s\to 0} G(s)$, $K_v = \lim_{s\to 0} sG(s)$ and $K_a = \lim_{s\to 0} s^2G(s)$. K is the time-constant gain, M_{LF} is the flat low-frequency level in dB (type 0), and $\omega_{0\,\text{dB}}$ is where the low-frequency line, extended if needed, crosses 0 dB.

RHP ZERO

$$(1 - j\omega T): \quad |\cdot| = \sqrt{1+\omega^2T^2}, \qquad \angle = -\tan^{-1}(\omega T)$$

Same magnitude as an LHP zero, but the phase is a lag. $T > 0$.

FIRST-ORDER ALL-PASS

$$\frac{1-sT}{1+sT}: \quad |\cdot| = 1, \qquad \angle = -2\tan^{-1}(\omega T)$$

$T > 0$. Magnitude is 0 dB everywhere and the phase goes from 0° to -180°, even though $n - m = 0$.

TIME DELAY

$$e^{-j\omega T}: \quad |\cdot| = 1, \qquad \angle = -\omega T \text{ rad} = -\frac{180^\circ}{\pi}\omega T$$

The magnitude plot is unchanged; the phase lag grows linearly with ω, which curves downward on a log-frequency axis. Remember 1 rad = 57.3°.

INTEGRATOR WITH DELAY

$$GH = \frac{Ke^{-sT}}{s}: \quad \omega_{pc} = \frac{\pi}{2T}, \qquad \text{GM} = \frac{\pi}{2KT}, \qquad \text{stable} \iff 0 < K < \frac{\pi}{2T}$$

Negative feedback, $K > 0$, $T > 0$. The phase crossover comes from $-90^\circ - \omega T \cdot \frac{180^\circ}{\pi} = -180^\circ$. Here $|GH| = K/\omega$ crosses 1 only once, at $\omega_{gc} = K$, and the phase falls monotonically, so the margins decide stability. Equivalently PM $= 90^\circ - KT \cdot \frac{180^\circ}{\pi}$ must be positive.

START AND END POINTS

$$\omega \to 0: \quad G \to K\angle 0^\circ \;\; (N = 0), \qquad G \to \infty\angle(-90^\circ N) \;\; (N \geq 1)$$
$$\omega \to \infty: \quad G \to 0\angle(-90^\circ(n-m)) \;\; (n > m)$$

For a minimum-phase function with $K > 0$ in time-constant form. The plot reaches the origin tangent to the axis at angle $-90^\circ(n-m)$. If $n = m$, it ends at the finite point equal to the ratio of the leading coefficients.

AXIS CROSSINGS

$$\text{Im}\,G(j\omega) = 0 \;\Rightarrow\; \text{real-axis crossing}; \qquad \text{Re}\,G(j\omega) = 0 \;\Rightarrow\; \text{imaginary-axis crossing}$$

A crossing of the **negative** real axis happens at the phase crossover frequency ω_{pc}. If the plot crosses at $-a$, then GM $= 1/a$.

LOW-FREQUENCY ASYMPTOTE OF A TYPE-1 POLAR PLOT

$$G(s) = \frac{K\prod(1+sT_z)}{s\prod(1+sT_p)}: \qquad \lim_{\omega\to 0}\operatorname{Re}G(j\omega) = -K\left(\sum T_p - \sum T_z\right)$$

The plot starts at infinity along −90°, running beside this vertical line. For $K/[s(1+sT)]$ the asymptote is $\mathrm{Re} = -KT$.

CROSSOVER FREQUENCIES

$$|G(j\omega_{gc})H(j\omega_{gc})| = 1 \;\; (0\text{ dB}), \qquad \angle G(j\omega_{pc})H(j\omega_{pc}) = -180^\circ$$

Both are open-loop quantities, found on the Bode plot or on the polar plot (unit circle and negative real axis).

GAIN MARGIN

$$\mathrm{GM} = \frac{1}{|G(j\omega_{pc})H(j\omega_{pc})|}, \qquad \mathrm{GM_{dB}} = -20\log_{10}|G(j\omega_{pc})H(j\omega_{pc})|$$

If the polar plot crosses the negative real axis at $-a$, then GM $= 1/a$. A positive dB value means stable, in the standard case described below.

PHASE MARGIN

$$\mathrm{PM} = 180^\circ + \angle G(j\omega_{gc})H(j\omega_{gc})$$

Use the phase as a negative angle: -150° gives PM $= 30^\circ$. On the polar plot, PM is the angle from the negative real axis to the point where the plot meets the unit circle.

LARGEST GAIN AND LARGEST DELAY

$$K_{\text{marginal}} = K\cdot\mathrm{GM}, \qquad T_{\max} = \frac{\mathrm{PM}\times\pi/180}{\omega_{gc}} \quad (\mathrm{PM\ in\ degrees})$$

K is the gain at which GM was computed. $T_{\max}$ is the delay that uses up the whole phase margin; it assumes a single gain crossover, which a delay does not move.

TYPE-1, THIRD-ORDER LOOP

$$GH = \frac{K}{s(1+sT_1)(1+sT_2)}: \quad \omega_{pc} = \frac{1}{\sqrt{T_1T_2}}, \quad |GH(j\omega_{pc})| = \frac{KT_1T_2}{T_1+T_2}, \quad K_{\text{marginal}} = \frac{T_1+T_2}{T_1T_2}$$

$T_1, T_2 > 0$. Comes from $\tan^{-1}a + \tan^{-1}b = 90^\circ \iff ab = 1$ for $a, b > 0$. It agrees with the Routh result for $T_1T_2s^3 + (T_1+T_2)s^2 + s + K = 0$. In pole-zero form $GH = K/[s(s+a)(s+b)]$ with $a, b > 0$: $\omega_{pc} = \sqrt{ab}$ and $K_{\text{marginal}} = ab(a+b)$.

The Nyquist stability criterion

PRINCIPLE OF THE ARGUMENT

$$N_0 = Z - P$$

The contour is traversed clockwise, encloses Z zeros and P poles of $F(s)$, and passes through none of them. N_0 = net number of clockwise encirclements of the ORIGIN by the image of $F(s)$. A counterclockwise encirclement counts as -1.

IMAGE OF THE INDENTATION AT THE ORIGIN

$$G(s)H(s) \approx \frac{K_r}{s^r} \text{ near } s = 0 \;\Rightarrow\; GH\left(\varepsilon e^{j\theta}\right) \approx \frac{K_r}{\varepsilon^r}e^{-jr\theta}, \quad \theta: -90^\circ \to +90^\circ$$

For a type-r loop with low-frequency gain constant K_r. The image angle runs from $+r\cdot 90^\circ$ (at $\omega = 0^-$) to $-r\cdot 90^\circ$ (at $\omega = 0^+$), with radius $\to\infty$ and always clockwise. If $K_r < 0$ (for example, when an open-loop RHP pole makes the low-frequency gain negative), the arc is rotated by 180° but still runs clockwise.

NYQUIST STABILITY CRITERION

$$Z = N + P$$

Convention: N = net number of CLOCKWISE encirclements of $-1 + j0$ by the Nyquist plot of $G(s)H(s)$ as s goes clockwise around the Nyquist contour. Each counterclockwise encirclement counts -1. P = number of open-loop poles (poles of GH) in the open RHP; $j\omega$-axis poles are left out by the indentation. Z = number of closed-loop poles in the RHP. Valid only if the plot does not pass through $-1 + j0$.

CONDITION FOR CLOSED-LOOP STABILITY

$$Z = 0 \;\Leftrightarrow\; N = -P$$

The closed loop is stable if and only if the Nyquist plot encircles $-1 + j0$ exactly P times COUNTERCLOCKWISE. For $P = 0$: stable if and only if $-1 + j0$ is not encircled at all ($N = 0$).

VARIABLE GAIN: MOVE THE CRITICAL POINT TO -1/K

$$1 + K\,G_1(s)H_1(s) = 0 \;\Leftrightarrow\; G_1(s)H_1(s) = -\frac{1}{K}$$

For $GH = KG_1H_1$ with $K > 0$, the plot of KG_1H_1 encircles -1 exactly as the plot of G_1H_1 encircles $-1/K$. Sketch G_1H_1 once, then slide the critical point along the negative real axis to get the range of K. For $K < 0$ the critical point $-1/K$ lies on the positive real axis instead.

START AND END OF THE POLAR PLOT

$$\lim_{\omega \to 0^+} \angle GH(j\omega) = -r \cdot 90^\circ, \qquad \lim_{\omega \to \infty} \angle GH(j\omega) = -(n - m) \cdot 90^\circ$$

For a positive gain, r poles at the origin, n poles and m zeros in total, and every other pole and zero in the LHP. As $\omega \to 0^+$ the magnitude tends to ∞ for $r \geq 1$ and to $GH(0) > 0$ for $r = 0$. As $\omega \to \infty$ it tends to 0 when $n > m$. Each RHP pole shifts the starting angle by -180° and each RHP zero by $+180^\circ$. The end angle is unchanged.

SHORTCUT: K/(S(S+A)(S+B))

$$GH = \frac{K}{s(s+a)(s+b)}: \quad \omega_{pc} = \sqrt{ab}, \quad GH(j\omega_{pc}) = -\frac{K}{ab(a+b)}, \quad K_{\max} = ab(a+b)$$

Holds for $a, b > 0$ with negative feedback. The plot crosses the negative real axis once. The closed loop is stable for $0 < K < ab(a+b)$ and oscillates at $\sqrt{ab}$ rad/s when $K = K_{\max}$. The low-frequency vertical asymptote is $\mathrm{Re} = -K(a+b)/(ab)^2$.

PHASE-CROSSOVER FREQUENCY

$$\angle G(j\omega_{pc})H(j\omega_{pc}) = -180^\circ \quad \Leftrightarrow \quad \mathrm{Im}\,[GH(j\omega_{pc})] = 0, \;\; \mathrm{Re}\,[GH(j\omega_{pc})] < 0$$

The frequency at which the polar plot cuts the negative real axis (the angle is taken modulo 360°).

GAIN MARGIN

$$\mathrm{GM} = \frac{1}{|G(j\omega_{pc})H(j\omega_{pc})|} = \frac{1}{a}, \qquad \mathrm{GM}_{\mathrm{dB}} = 20\log_{10}\frac{1}{a} = -20\log_{10} a$$

Here the plot crosses the negative real axis at $-a$ ($a > 0$). GM is the factor by which the loop gain can be multiplied before that crossing reaches -1. If the plot never reaches -180° at a finite frequency (no phase crossover), GM $= \infty$.

GAIN-CROSSOVER FREQUENCY AND PHASE MARGIN

$$|G(j\omega_{gc})H(j\omega_{gc})| = 1, \qquad \mathrm{PM} = 180^\circ + \angle G(j\omega_{gc})H(j\omega_{gc})$$

ω_{gc} is where the plot cuts the unit circle. Measure the angle as a negative number between 0° and -360°: $\angle GH = -140^\circ$ gives PM $= 40^\circ$. PM is the extra phase lag at ω_{gc} that would swing that point onto -1. If $|GH| < 1$ at every frequency, there is no gain crossover and PM is taken as ∞.

LARGEST TOLERABLE PURE DELAY

$$T_{\max} = \frac{\text{PM (in rad)}}{\omega_{gc}}$$

A delay e^{-sT} has $|e^{-j\omega T}| = 1$ and phase $-\omega T$; on its own it traces the unit circle clockwise. It rotates every point of the plot clockwise by ωT without changing its magnitude, so at ω_{gc} it uses up phase margin. Valid when the delay-free closed loop is stable with PM > 0 and the plot cuts the unit circle only once.

PHASE MARGIN OF THE STANDARD SECOND-ORDER LOOP

$$G(s) = \frac{\omega_n^2}{s(s+2\zeta\omega_n)}: \quad \omega_{gc} = \omega_n\sqrt{\sqrt{1+4\zeta^4} - 2\zeta^2}, \quad \text{PM} = \tan^{-1}\frac{2\zeta}{\sqrt{\sqrt{1+4\zeta^4} - 2\zeta^2}}$$

Unity negative feedback with $\zeta > 0$, so the closed loop is $\frac{\omega_n^2}{s^2+2\zeta\omega_n s+\omega_n^2}$. PM depends on ζ alone. Example: $\zeta = 0.5$ gives PM $\approx 51.8^\circ$. As a rough check only, PM in degrees is close to 100ζ for $0 < \zeta \le 0.6$ (within about 3.5°). Use the exact formula for a numerical answer.

Compensators and PID controllers

ERROR CONSTANTS AND STEADY-STATE ERROR

$$K_{\text{pos}} = \lim_{s\to 0} G_c(s)G(s), \quad K_v = \lim_{s\to 0} sG_c(s)G(s), \quad K_a = \lim_{s\to 0} s^2 G_c(s)G(s)$$

$$e_{ss,\text{step}} = \frac{A}{1+K_{\text{pos}}}, \quad e_{ss,\text{ramp}} = \frac{A}{K_v}, \quad e_{ss,\text{parabola}} = \frac{A}{K_a}$$

Applies to unity feedback with a stable closed loop, for inputs $A\,u(t)$, $At\,u(t)$ and $\frac{A}{2}t^2u(t)$. A type-N loop (N poles at the origin) has zero error for inputs of lower order than t^N, a finite error for t^N, and an infinite error for higher orders.

CONVERTING A DAMPING SPECIFICATION INTO A PHASE MARGIN

$$\text{PM} = \tan^{-1}\frac{2\zeta}{\sqrt{\sqrt{1+4\zeta^4} - 2\zeta^2}}, \qquad \omega_{gc} = \omega_n\sqrt{\sqrt{1+4\zeta^4} - 2\zeta^2}, \qquad \text{PM} \approx 100\,\zeta \text{ degrees}$$

Exact for the unity-feedback loop $\frac{\omega_n^2}{s(s+2\zeta\omega_n)}$, whose closed loop is a pure second-order system. The straight-line rule $\text{PM} \approx 100\zeta$ degrees stays within about 3.5° of the exact value for $0 < \zeta \le 0.6$ and drifts off above that ($\zeta = 0.7$ gives 65.2°, not 70°). For a higher-order loop it is only an estimate, valid when one complex pole pair dominates. Lead and lag design from the Bode plot start by turning a ζ or overshoot specification into a required PM this way.

PID TRANSFER FUNCTION

$$G_c(s) = K_p + \frac{K_i}{s} + K_d s = K_p\left(1 + \frac{1}{T_i s} + T_d s\right) = \frac{K_d s^2 + K_p s + K_i}{s}$$

Has one pole at the origin and two zeros, the roots of $K_d s^2 + K_p s + K_i = 0$. The zeros are real when $K_p^2 \ge 4K_iK_d$ (equivalently $T_i \ge 4T_d$) and a complex pair otherwise. With $K_i \neq 0$ it raises the system type by 1.

PI CONTROLLER

$$G_c(s) = K_p + \frac{K_i}{s} = K_p\left(1 + \frac{1}{T_i s}\right) = \frac{K_p\left(s + \frac{1}{T_i}\right)}{s}$$

Assumes $K_p > 0$ and $T_i > 0$. Pole at $s = 0$ and zero at $s = -1/T_i = -K_i/K_p$. Phase $= -\tan^{-1}\dfrac{1}{\omega T_i}$, which lies between -90° and 0°. Raises the type by 1.

PD CONTROLLER

$$G_c(s) = K_p + K_d s = K_p(1 + T_d s)$$

Assumes $K_p > 0$ and $T_d > 0$. Zero at $s = -1/T_d = -K_p/K_d$ and no pole. Phase $= +\tan^{-1}(\omega T_d)$, which lies between 0° and $+90^\circ$. The system type does not change.

DERIVATIVE (PD) CONTROL OF A SECOND-ORDER PLANT

$$G(s) = \frac{\omega_n^2}{s(s+2\zeta\omega_n)}, \quad G_c(s) = 1 + T_d s: \qquad \zeta' = \zeta + \frac{\omega_n T_d}{2}$$

The closed-loop characteristic equation becomes $s^2 + (2\zeta\omega_n + \omega_n^2 T_d)s + \omega_n^2 = 0$. So ω_n and $K_v = \omega_n/(2\zeta)$ are unchanged and the damping increases. The closed loop also gains a zero at $s = -1/T_d$. For an underdamped ζ' this zero makes the response rise faster and overshoot more than a pure second-order system with the same ζ', and the effect is stronger the closer the zero lies to the origin compared with $\zeta'\omega_n$.

RATE (TACHOMETER) FEEDBACK, FOR COMPARISON

$$G(s) = \frac{\omega_n^2}{s(s+2\zeta\omega_n)} \text{ with minor loop } K_t s: \quad \zeta' = \zeta + \frac{K_t\,\omega_n}{2}, \quad K_v = \frac{\omega_n}{2\zeta + K_t\,\omega_n}$$

The inner loop reduces to $\dfrac{\omega_n^2}{s(s+2\zeta\omega_n + K_t\omega_n^2)}$, so the characteristic equation is the same as with PD and $T_d = K_t$, and there is no closed-loop zero. But K_v falls below $\omega_n/(2\zeta)$, so the ramp error grows. PD in the forward path keeps K_v unchanged.

OP-AMP PI AND PD CONTROLLERS (INVERTING STAGE)

$$\text{PI: } Z_1 = R_1,\ Z_2 = R_2 + \frac{1}{C_2 s}: \quad \frac{V_o}{V_i} = -\frac{R_2}{R_1}\left(1 + \frac{1}{R_2 C_2 s}\right)$$

$$\text{PD: } Z_1 = R_1 \parallel C_1,\ Z_2 = R_2: \quad \frac{V_o}{V_i} = -\frac{R_2}{R_1}\left(1 + R_1 C_1 s\right)$$

Ideal op-amp, $\dfrac{V_o}{V_i} = -\dfrac{Z_2}{Z_1}$ with Z_1 the input impedance and Z_2 the feedback impedance. PI: $K_p = R_2/R_1$ and $T_i = R_2 C_2$. PD: $K_p = R_2/R_1$ and $T_d = R_1 C_1$. A second inverting stage of gain $-R_4/R_3$ removes the minus sign.

LEAD COMPENSATOR (TIME-CONSTANT FORM)

$$G_c(s) = \frac{1 + aTs}{1 + Ts}, \quad a > 1; \qquad \text{zero at } -\frac{1}{aT}, \quad \text{pole at } -\frac{1}{T}$$

Corner frequencies are $\frac{1}{aT}$ (zero) and $\frac{1}{T}$ (pole). The gain rises from 1 (0 dB) at low frequency to a ($20\log_{10} a$ dB) at high frequency, which is a high-pass shape.

MAXIMUM PHASE LEAD

$$\sin\phi_m = \frac{a-1}{a+1}, \qquad \tan\phi_m = \frac{a-1}{2\sqrt{a}}, \qquad a = \frac{1+\sin\phi_m}{1-\sin\phi_m}$$

Valid for the time-constant form with $a > 1$. ϕ_m depends only on the ratio a of the two corner frequencies, not on T. For any finite a, $\phi_m < 90^\circ$.

FREQUENCY OF MAXIMUM PHASE LEAD

$$\omega_m = \frac{1}{T\sqrt{a}} = \sqrt{\frac{1}{aT}\cdot\frac{1}{T}}$$

The geometric mean of the two corner frequencies, which is their midpoint on a log scale.

GAIN AT THE FREQUENCY OF MAXIMUM LEAD

$$|G_c(j\omega_m)| = \sqrt{a} \quad \text{i.e. } 10\log_{10} a \text{ dB}$$

Holds for the unity-DC-gain form. In lead design, ω_m is placed at the new gain crossover, where the uncompensated magnitude is $-10\log_{10} a$ dB.

OTHER FORMS OF THE LEAD COMPENSATOR

$$\frac{1+Ts}{1+\alpha Ts},\ 0<\alpha<1: \quad \sin\phi_m=\frac{1-\alpha}{1+\alpha}, \quad \omega_m=\frac{1}{T\sqrt{\alpha}}$$

$$\frac{s+z}{s+p},\ 0<z<p: \quad \sin\phi_m=\frac{p-z}{p+z}, \quad \omega_m=\sqrt{zp}, \quad |G_c(j\omega_m)|=\sqrt{\frac{z}{p}}$$

$\alpha = 1/a$. The pole-zero form has DC gain $z/p < 1$ and high-frequency gain 1. Its phase is identical to the time-constant form with the same corners.

PASSIVE RC LEAD NETWORK

$$\frac{V_o(s)}{V_i(s)}=\frac{R_2(1+R_1Cs)}{R_1+R_2+R_1R_2Cs}=\frac{s+\frac{1}{R_1C}}{s+\frac{1}{R_1C}+\frac{1}{R_2C}}=\frac{1}{a}\cdot\frac{1+aTs}{1+Ts}$$

Series arm: R_1 in parallel with C. Shunt arm: R_2 across the output. Assumes an ideal source and an unloaded output. $a=\dfrac{R_1+R_2}{R_2}$, $aT=R_1C$, $T=\dfrac{R_1R_2C}{R_1+R_2}$. The network attenuates DC by $1/a$, so an amplifier of gain a is needed to keep the error constant.

LAG COMPENSATOR (TIME-CONSTANT FORM)

$$G_c(s)=\frac{1+Ts}{1+bTs},\quad b>1; \qquad \text{zero at } -\frac{1}{T}, \quad \text{pole at } -\frac{1}{bT}$$

High-frequency gain is $1/b$, an attenuation of $20\log_{10} b$ dB above the zero corner.

MAXIMUM PHASE LAG

$$\sin\phi_m=-\frac{b-1}{b+1} \quad \text{at} \quad \omega_m=\frac{1}{T\sqrt{b}}$$

Same algebra as the lead, with the sign reversed. ω_m is the geometric mean of the corners $\frac{1}{bT}$ and $\frac{1}{T}$.

LAG IN POLE-ZERO FORM AND ITS EFFECT ON STEADY-STATE ERROR

$$G_c(s)=\frac{s+z}{s+p}, \quad z>p>0: \qquad K_v^{\text{new}}=\frac{z}{p}\,K_v^{\text{old}} \quad (\text{likewise } K_{\text{pos}},\ K_a)$$

Holds at the same loop gain K, for a stable compensated loop. DC gain is z/p and high-frequency gain is 1. When z and p both sit close to the origin (a dipole), the angle they add at the dominant poles is small. The transient response then barely changes while the error constant is multiplied by z/p.

PASSIVE RC LAG NETWORK

$$\frac{V_o(s)}{V_i(s)}=\frac{1+R_2Cs}{1+(R_1+R_2)Cs}, \qquad T=R_2C, \quad b=\frac{R_1+R_2}{R_2}$$

Series arm: R_1. Shunt arm: R_2 in series with C, with the output taken across this shunt arm. Assumes an unloaded output. DC gain is 1, so no amplifier is needed to hold the low-frequency gain.

OP-AMP LEAD OR LAG NETWORK

$$\frac{V_o}{V_i}=\frac{R_2R_4}{R_1R_3}\cdot\frac{1+R_1C_1s}{1+R_2C_2s} \qquad (Z_1=R_1\parallel C_1,\ Z_2=R_2\parallel C_2,\ \text{then an inverter of gain } -R_4/R_3)$$

Ideal op-amps. The first inverting stage gives $-\dfrac{R_2}{R_1}\cdot\dfrac{1+R_1C_1s}{1+R_2C_2s}$ and the second stage cancels the sign. Zero at $-1/(R_1C_1)$, pole at $-1/(R_2C_2)$: a **lead** if $R_1C_1>R_2C_2$ and a **lag** if $R_1C_1<R_2C_2$. Unlike the passive networks, the DC gain $\dfrac{R_2R_4}{R_1R_3}$ can be set independently of the pole-zero ratio.

LEAD-LAG COMPENSATOR

$$G_c(s)=\frac{1+T_2s}{1+bT_2s}\cdot\frac{1+aT_1s}{1+T_1s}, \qquad a>1,\ b>1, \quad \frac{1}{T_2}\ll\frac{1}{aT_1}$$

The first factor is the lag section and the second is the lead section. Both lag corners lie well below both lead corners.

PASSIVE LAG-LEAD NETWORK (SINGLE RATIO)

$$G_c(s) = \frac{(1+T_1 s)(1+T_2 s)}{\left(1+\frac{T_1}{\beta}s\right)(1+\beta T_2 s)}, \quad \beta > 1,\ T_2 > T_1; \qquad \angle G_c = 0 \text{ at } \omega_0 = \frac{1}{\sqrt{T_1 T_2}}$$

Realised by one RC network: series arm R_1 in parallel with C_1, shunt arm R_2 in series with C_2, output across the shunt arm, unloaded. Then $T_1 = R_1C_1$ and $T_2 = R_2C_2$, and the denominator $1 + (R_1C_1 + R_2C_2 + R_1C_2)s + R_1C_1R_2C_2s^2$ factors as $\left(1+\frac{T_1}{\beta}s\right)(1+\beta T_2 s)$, so the lead and lag sections are forced to share the same ratio β. The phase is a lag below ω_0 and a lead above it. DC and high-frequency gains are both 1, with a dip in between.

ZIEGLER-NICHOLS PID IN POLE-ZERO FORM

$$\text{Step-response method } (K=1): \quad G_c(s) = 0.6\,T\,\frac{\left(s+\frac{1}{L}\right)^2}{s}$$

$$\text{Ultimate-gain method:} \quad G_c(s) = 0.075\,K_u P_u\,\frac{\left(s+\frac{4}{P_u}\right)^2}{s}$$

Both PID rows set $T_i = 4T_d$, which is exactly the condition $K_p^2 = 4K_iK_d$ for a double zero. So both give a pole at the origin and a double zero on the negative real axis, at $-1/L$ and at $-4/P_u$ respectively. The leading coefficient is K_d, which makes a quick check. For a process gain $K \neq 1$, divide the step-response result by K.

State-space models

STATE MODEL OF A CONTINUOUS-TIME LTI SYSTEM

$$\dot{x}(t) = A\,x(t) + B\,u(t) \qquad \text{(state equation)}$$
$$y(t) = C\,x(t) + D\,u(t) \qquad \text{(output equation)}$$

n states, m inputs, p outputs. A, B, C, D are constant matrices because the system is time-invariant.

CONTROLLABLE CANONICAL (PHASE-VARIABLE) FORM

$$A = \begin{bmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \\ -a_0 & -a_1 & -a_2 \end{bmatrix}, \quad B = \begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix}, \quad C = \begin{bmatrix} b_0 & b_1 & b_2 \end{bmatrix}, \quad D = 0$$

The denominator must be monic. The last row of A holds the negated denominator coefficients in **ascending** powers of s, and C holds the numerator coefficients in ascending powers. This form is always controllable. Some texts number the states in reverse, which puts the coefficients in the first row; $G(s)$ is the same either way.

OBSERVABLE CANONICAL FORM (DUAL OF THE CONTROLLABLE FORM)

$$A = \begin{bmatrix} 0 & 0 & -a_0 \\ 1 & 0 & -a_1 \\ 0 & 1 & -a_2 \end{bmatrix}, \quad B = \begin{bmatrix} b_0 \\ b_1 \\ b_2 \end{bmatrix}, \quad C = \begin{bmatrix} 0 & 0 & 1 \end{bmatrix}, \quad D = 0$$

Convention used here: $A_o = A_c^T$, $B_o = C_c^T$, $C_o = B_c^T$. This form is always observable. Reversing the state order flips rows and columns without changing $G(s)$.

DIAGONAL (PARALLEL) FORM: DISTINCT POLES

$$G(s) = d + \sum_{i=1}^{3} \frac{r_i}{s-\lambda_i} \;\Rightarrow\; A = \begin{bmatrix} \lambda_1 & 0 & 0 \\ 0 & \lambda_2 & 0 \\ 0 & 0 & \lambda_3 \end{bmatrix},\ B = \begin{bmatrix} 1 \\ 1 \\ 1 \end{bmatrix},\ C = \begin{bmatrix} r_1 & r_2 & r_3 \end{bmatrix},\ D = d$$

Only for distinct poles; the r_i are the partial-fraction residues. The residues can sit in B instead of C, because only each product $b_ic_i = r_i$ is fixed. Complex poles give complex entries.

TRANSFER FUNCTION FROM A STATE MODEL

$$G(s) = \frac{Y(s)}{U(s)} = C\,(sI - A)^{-1}B + D = \frac{C\,\mathrm{adj}(sI - A)\,B}{\det(sI - A)} + D$$

Assumes zero initial conditions. For MIMO systems $G(s)$ is a $p \times m$ matrix with $G_{ij} = Y_i/U_j$. The order of multiplication is fixed: C, then $(sI - A)^{-1}$, then B.

CHARACTERISTIC EQUATION

$$\det(sI - A) = 0$$

Its roots are the eigenvalues of A. Every pole of $G(s)$ is an eigenvalue. If $C\,\mathrm{adj}(sI - A)\,B$ shares a factor with $\det(sI - A)$, that eigenvalue cancels and does not appear as a pole. $\det(A - \lambda I)$ differs from $\det(\lambda I - A)$ only by $(-1)^n$, so the two have the same roots.

INVERSE OF A 2×2 MATRIX

$$\begin{bmatrix} a & b \\ c & d \end{bmatrix}^{-1} = \frac{1}{ad - bc}\begin{bmatrix} d & -b \\ -c & a \end{bmatrix}, \qquad ad - bc \neq 0$$

Swap the diagonal entries, negate the off-diagonal ones and divide by the determinant. You use this in nearly every 2×2 $(sI - A)^{-1}$ calculation.

EIGENVALUE SHORTCUTS

$$\sum_i \lambda_i = \mathrm{tr}(A), \qquad \prod_i \lambda_i = \det A, \qquad n = 2:\ s^2 - \mathrm{tr}(A)\,s + \det A = 0$$

Holds for any square A, with eigenvalues counted by multiplicity. The eigenvalues of a triangular or diagonal matrix are its diagonal entries.

SIMILARITY TRANSFORMATION

$$x = Pz \ \Rightarrow\ \bar{A} = P^{-1}AP, \quad \bar{B} = P^{-1}B, \quad \bar{C} = CP, \quad \bar{D} = D$$

P is any invertible $n \times n$ matrix. The transformation does not change the eigenvalues, the characteristic equation, $G(s)$, controllability or observability. With P = modal matrix (eigenvectors as columns), $\bar{A}$ is diagonal. This needs n linearly independent eigenvectors, which is guaranteed when the eigenvalues are distinct.

STATE TRANSITION MATRIX

$$\Phi(t) = e^{At} = \sum_{k=0}^{\infty} \frac{A^k t^k}{k!} = I + At + \frac{A^2t^2}{2!} + \cdots = \mathcal{L}^{-1}\left[(sI - A)^{-1}\right]$$

Valid for any constant square A. The series stops after finitely many terms if A is nilpotent, e.g. $A = \begin{bmatrix} 0 & 1 \\ 0 & 0 \end{bmatrix}$ gives $e^{At} = \begin{bmatrix} 1 & t \\ 0 & 1 \end{bmatrix}$.

PROPERTIES OF THE STM

$$\begin{aligned}
&\Phi(0) = I \\
&\Phi^{-1}(t) = \Phi(-t) \\
&\Phi(t_1 + t_2) = \Phi(t_1)\,\Phi(t_2) \\
&\Phi(t_2 - t_0) = \Phi(t_2 - t_1)\,\Phi(t_1 - t_0) \\
&[\Phi(t)]^k = \Phi(kt) \quad (k \text{ any integer}) \\
&\dot{\Phi}(t) = A\,\Phi(t) = \Phi(t)\,A, \qquad A = \dot{\Phi}(0) \\
&\det \Phi(t) = e^{\mathrm{tr}(A)\,t} \neq 0
\end{aligned}$$

True for every constant A, so $\Phi(t)$ is always invertible. Use $\Phi(0) = I$ and $\dot{\Phi}(0) = A$ to test whether a given matrix can be an STM, and to recover A from a given STM.

STMS TO RECOGNISE ON SIGHT

$$A = \begin{bmatrix} \lambda_1 & 0 \\ 0 & \lambda_2 \end{bmatrix} \Rightarrow e^{At} = \begin{bmatrix} e^{\lambda_1 t} & 0 \\ 0 & e^{\lambda_2 t} \end{bmatrix}$$

$$A = \begin{bmatrix} \lambda & 1 \\ 0 & \lambda \end{bmatrix} \Rightarrow e^{At} = e^{\lambda t} \begin{bmatrix} 1 & t \\ 0 & 1 \end{bmatrix}$$

$$A = \begin{bmatrix} \sigma & \omega \\ -\omega & \sigma \end{bmatrix} \Rightarrow e^{At} = e^{\sigma t} \begin{bmatrix} \cos\omega t & \sin\omega t \\ -\sin\omega t & \cos\omega t \end{bmatrix}$$

$$A = P\Lambda P^{-1} \Rightarrow e^{At} = P\, e^{\Lambda t}\, P^{-1}$$

The third matrix has eigenvalues $\sigma \pm j\omega$. The last line is the diagonalisation method: Λ is the diagonal matrix of eigenvalues and P is the modal matrix, which exists when A has n linearly independent eigenvectors (always the case for distinct eigenvalues).

COMPLETE SOLUTION OF THE STATE EQUATION

$$x(t) = \underbrace{e^{At}x(0)}_{\text{zero-input}} + \underbrace{\int_0^t e^{A(t-\tau)} B\, u(\tau)\, d\tau}_{\text{zero-state}}$$

Initial time 0. For initial time t_0: $x(t) = \Phi(t-t_0)x(t_0) + \int_{t_0}^{t} \Phi(t-\tau)Bu(\tau)\,d\tau$.

SOLUTION IN THE S-DOMAIN

$$X(s) = (sI - A)^{-1}x(0) + (sI - A)^{-1}B\,U(s), \qquad Y(s) = C\,X(s) + D\,U(s)$$

Often the quickest route by hand. Take the inverse transform of each entry at the end.

UNIT-STEP RESPONSE AND FINAL VALUES

$$x_{zs}(t) = A^{-1}\left(e^{At} - I\right)B, \qquad x(\infty) = -A^{-1}B, \qquad y(\infty) = -CA^{-1}B + D = G(0)$$

The first formula (unit-step input, zero initial state) needs A invertible. The final values hold, for any initial state, when every eigenvalue of A has a negative real part; A is then automatically invertible, because 0 is not an eigenvalue. You can also get $x(\infty)$ by setting $\dot{x} = 0$: $0 = Ax + B$.

DISCRETE-TIME STATE MODEL AND ITS SOLUTION

$$x[k+1] = A\,x[k] + B\,u[k], \qquad y[k] = C\,x[k] + D\,u[k]$$

$$x[k] = A^k x[0] + \sum_{i=0}^{k-1} A^{k-1-i} B\, u[i]$$

$$A^k = \mathcal{Z}^{-1}\left[z\,(zI - A)^{-1}\right], \qquad G(z) = C\,(zI - A)^{-1}B + D$$

Here A, B, C, D are the discrete-time matrices, and A^k plays the role of the STM. The system is asymptotically stable exactly when every eigenvalue of A has $|\lambda| < 1$. The Kalman matrices $[B \;\; AB \;\; \cdots \;\; A^{n-1}B]$ and $[C;\; CA;\; \cdots;\; CA^{n-1}]$ take the same form: full rank of the first means every state can be reached from the origin, and full rank of the second means $x[0]$ can be recovered from the output. Sampling $\dot{x} = Ax + Bu$ with a zero-order hold of period T gives $A_d = e^{AT}$ and $B_d = \left(\int_0^T e^{A\sigma}\,d\sigma\right)B$, which equals $A^{-1}(e^{AT} - I)B$ when A is invertible.

KALMAN CONTROLLABILITY TEST

$$Q_c = \begin{bmatrix} B & AB & A^2B & \cdots & A^{n-1}B \end{bmatrix}, \qquad \text{controllable} \iff \operatorname{rank} Q_c = n$$

Depends only on (A, B). For a single input, Q_c is $n \times n$ and the system is controllable exactly when $\det Q_c \neq 0$. For m inputs, Q_c is $n \times nm$ and you must check its rank. Compute A^2B as $A(AB)$.

KALMAN OBSERVABILITY TEST

$$Q_o = \begin{bmatrix} C \\ CA \\ CA^2 \\ \vdots \\ CA^{n-1} \end{bmatrix}, \qquad \text{observable} \iff \operatorname{rank} Q_o = n$$

Depends only on (A, C). An equivalent transposed convention is $[C^T \;\; A^TC^T \;\; \cdots \;\; (A^T)^{n-1}C^T]$, which has the same rank. For a single output, the system is observable exactly when $\det Q_o \neq 0$.

DIAGONAL-FORM TEST

$\bar{A} = \operatorname{diag}(\lambda_1, \ldots, \lambda_n)$ distinct : $\quad$ controllable $\iff$ no zero row in $\bar{B}$, $\qquad$ observable $\iff$ no zero column in $\bar{C}$

Use only after transforming to a diagonal $\bar{A}$ with distinct eigenvalues. A zero row i of $\bar{B}$ means the mode $e^{\lambda_i t}$ cannot be excited, and a zero column i of $\bar{C}$ means that mode never reaches the output. In Jordan form (one block per distinct eigenvalue), check the row of $\bar{B}$ at the **last** row of each block and the column of $\bar{C}$ at the **first** column of each block. The test does not apply to a diagonal $\bar{A}$ with a repeated eigenvalue; see the last tip of this section.

STATE FEEDBACK AND POLE PLACEMENT

$u = -Kx + r \;\Rightarrow\; \dot{x} = (A - BK)\,x + B\,r, \qquad$ closed-loop characteristic equation: $\det(sI - A + BK) = 0$

single input: $K = \begin{bmatrix} 0 & \cdots & 0 & 1 \end{bmatrix} Q_c^{-1}\, \alpha_d(A)$

K is $m \times n$ ($1 \times n$ for a single input) and r is the new reference input. The eigenvalues of $A - BK$ can be placed at any chosen locations (complex ones in conjugate pairs) if and only if (A, B) is controllable. In the pole-placement formula, $\alpha_d(s) = s^n + \alpha_{n-1}s^{n-1} + \cdots + \alpha_0$ is the desired closed-loop characteristic polynomial and $\alpha_d(A) = A^n + \alpha_{n-1}A^{n-1} + \cdots + \alpha_0 I$. State feedback does not change controllability and, for a SISO system, does not move the zeros of the transfer function; it can destroy observability if a closed-loop pole is placed on a zero.

Actuators, time delay and industrial control

FREQUENCY RESPONSE OF A PURE DELAY

$$\left|e^{-j\omega T}\right| = 1, \qquad \angle e^{-j\omega T} = -\omega T \text{ rad} = -\frac{180^\circ}{\pi}\,\omega T$$

Holds at every ω (ω in rad/s, T in s). The magnitude plot is untouched. The phase lag grows linearly with ω, so on a logarithmic frequency axis it falls away more and more steeply.

PHASE MARGIN WITH AN ADDED DELAY

$$PM_T = PM_0 - \frac{180^\circ}{\pi}\,\omega_{gc}T, \qquad PM_0 = 180^\circ + \angle G(j\omega_{gc})H(j\omega_{gc})$$

PM_0 is the margin of the delay-free loop and ω_{gc} is its gain-crossover frequency (where $|GH| = 1$). This is valid because the delay leaves ω_{gc} where it was. Read stability from $PM_T > 0$ only when the magnitude crosses 0 dB once and the open loop has no right-half-plane poles.

DELAY MARGIN (LARGEST TOLERABLE DEAD TIME)

$$T_{\max} = \frac{PM_0 \text{ (in rad)}}{\omega_{gc}}$$

PM_0 is the phase margin of the delay-free loop, so $T_{\max}$ is the TOTAL loop delay at which the phase margin reaches zero and the closed loop becomes marginally stable. Convert the margin to radians first. If the loop already contains a delay T_0, the extra delay it can tolerate is $T_{\max} - T_0$, which equals (present phase margin in rad)$/\omega_{gc}$. Assumes the delay-free closed loop is stable ($PM_0 > 0$) and the magnitude crosses 0 dB only once.

INTEGRATOR PLUS DEAD TIME

$$G(s)H(s) = \frac{K\,e^{-sT}}{s} \text{ is closed-loop stable } \iff 0 < KT < \frac{\pi}{2}$$

Negative feedback, $K > 0$, $T > 0$. The gain crossover is at $\omega_{gc} = K$, where the phase margin is $\pi/2 - KT$ rad. The phase crossover is where $\pi/2 + \omega T = \pi$, i.e. $\omega_{pc} = \pi/(2T)$, and there $|GH| = 2KT/\pi$. Equivalently, the delay margin is $T_{\max} = \pi/(2K)$.

ULTIMATE GAIN AND FREQUENCY WITH DEAD TIME

$$\angle G_0(j\omega_u) - \omega_u T = -\pi, \qquad K_u = \frac{1}{|G_0(j\omega_u)|}$$

For the loop $K\,G_0(s)\,e^{-sT}$ under negative feedback, $K > 0$. Solve the phase condition (usually by trial) for its lowest root ω_u in rad/s, then read off K_u: the gain at which the closed loop is marginally stable and oscillates at ω_u. Valid when G_0 has no right-half-plane poles and $|G_0(j\omega)|$ decreases with frequency. The Pade example in the next section works one out exactly.

FIRST-ORDER-PLUS-DEAD-TIME (FOPDT) MODEL

$$G(s) = \frac{K\,e^{-\theta s}}{\tau s + 1}, \qquad \angle G(j\omega) = -\tan^{-1}(\omega\tau) - \omega\theta \text{ rad}$$

K = steady-state gain, τ = time constant, θ = dead time. The ratio θ/τ measures how hard the process is to control: the larger it is, the smaller the loop gain KK_c at which a proportional loop becomes unstable.

FOPDT STEP RESPONSE (INPUT STEP OF SIZE A)

$$y(t) = 0 \;\; (t < \theta), \qquad y(t) = KA\left(1 - e^{-(t-\theta)/\tau}\right) \;\; (t \geq \theta)$$

Nothing happens for θ seconds. After that the response is an ordinary first-order rise, reaching 63.2% of KA at $t = \theta + \tau$.

FIRST-ORDER PADE APPROXIMATION

$$e^{-sT} \approx \frac{1 - \dfrac{sT}{2}}{1 + \dfrac{sT}{2}} = \frac{2 - sT}{2 + sT}$$

An all-pass: magnitude exactly 1, phase $-2\tan^{-1}(\omega T/2)$. It puts a right-half-plane zero at $s = +2/T$ (non-minimum phase) and a pole at $s = -2/T$. Accurate while ωT is well below 1.

SECOND-ORDER PADE APPROXIMATION

$$e^{-sT} \approx \frac{1 - \dfrac{sT}{2} + \dfrac{s^2T^2}{12}}{1 + \dfrac{sT}{2} + \dfrac{s^2T^2}{12}}$$

Also all-pass, with two right-half-plane zeros (a complex pair). It follows the true phase to higher frequencies than the first-order form.

INHERENT CHARACTERISTICS (CONSTANT PRESSURE DROP ACROSS THE VALVE)

$$\text{Linear: } f = x \qquad \text{Equal percentage: } f = R^{x-1} \qquad \text{Quick opening: } f \approx \sqrt{x}$$

R is the rangeability. The ideal equal-percentage law gives $f = 1/R$ at $x = 0$, so it never reaches zero; the real valve shuts by seating. The square-root law is one common model of quick opening.

EQUAL-PERCENTAGE GAIN

$$\frac{df}{dx} = f \ln R \;\Rightarrow\; \frac{1}{f}\frac{df}{dx} = \ln R, \qquad \frac{f(x + \Delta x)}{f(x)} = R^{\Delta x}$$

Equal steps of travel change the current flow by equal percentages. The gain is proportional to flow: small near closed, large near fully open.

RANGEABILITY

$$R = \frac{q_{\max}}{q_{\min}} \quad (\text{controllable flows})$$

Ratio of the largest to the smallest flow the valve can control. For the ideal equal-percentage valve, $R = f(1)/f(0)$.

VALVE FLOW COEFFICIENT (LIQUIDS)

$$q = C_v\, f(x) \sqrt{\frac{\Delta P}{G}}$$

q in US gal/min, ΔP across the valve in psi, G = specific gravity relative to water. C_v is the water flow in gal/min through the fully open valve ($f = 1$) at a 1 psi drop. The metric coefficient K_v uses m³/h and bar, with $C_v \approx 1.156\, K_v$.

VALVE-CONTROLLED HYDRAULIC PISTON

$$q_L = A\frac{dy}{dt} \;\Rightarrow\; \frac{Y(s)}{X_v(s)} = \frac{K_q}{A\,s}$$

Piston area A. Neglects the load-pressure term $K_c p_L$ (no load), oil compressibility and leakage. The actuator is then an integrator, so a unity-feedback position loop with a proportional controller is type 1 and has zero steady-state error to a step command.

ARMATURE-CONTROLLED DC MOTOR

$$\frac{\theta(s)}{V_a(s)} = \frac{K_t}{s\left[(R_a + L_a s)(J s + B) + K_t K_b\right]}$$

K_t = torque constant (N·m/A), K_b = back-emf constant (V·s/rad); the two are numerically equal in SI units. J and B are the total inertia and viscous friction referred to the motor shaft. Speed is $\omega(s) = s\,\theta(s)$.

ARMATURE CONTROL, ARMATURE INDUCTANCE NEGLECTED

$$\frac{\theta(s)}{V_a(s)} = \frac{K_m}{s(\tau_m s + 1)}, \qquad K_m = \frac{K_t}{R_a B + K_t K_b}, \qquad \tau_m = \frac{R_a J}{R_a B + K_t K_b}$$

K_m in (rad/s)/V is the steady-state speed per volt. Back emf shortens the time constant from J/B to τ_m.

FIELD-CONTROLLED DC MOTOR

$$\frac{\theta(s)}{V_f(s)} = \frac{K_{tf}}{s\,(R_f + L_f s)(J s + B)}$$

Armature current is constant and the torque is $T = K_{tf}\, i_f$. With no back-emf feedback there is no electrical damping, and the field time constant L_f/R_f is significant. Used for small motors.

TWO-PHASE AC SERVO MOTOR

$$T = K_1 e_c - K_2\,\omega, \qquad \frac{\theta(s)}{E_c(s)} = \frac{K_m}{s(\tau_m s + 1)}, \qquad K_m = \frac{K_1}{B + K_2}, \quad \tau_m = \frac{J}{B + K_2}$$

The reference winding has a fixed voltage, and the control-winding voltage e_c is 90° out of phase with it. K_1 = stall torque per volt of e_c. K_2 = magnitude of the slope of the torque–speed line (stall torque ÷ no-load speed), which acts like extra viscous friction. The rotor has a high resistance-to-reactance ratio so that this line slopes downward over the whole speed range.

STEP ANGLE, VARIABLE RELUCTANCE (FULL STEP, ONE PHASE ON)

$$\beta = \frac{360^\circ}{m\,N_r} = \frac{N_s - N_r}{N_s\,N_r} \times 360^\circ$$

m = number of stator phases, N_r = rotor teeth, N_s = stator teeth (poles). The two forms agree for the usual designs, where $N_s - N_r = N_s/m$.

STEP ANGLE, HYBRID

$$\beta = \frac{360^\circ}{2\,m\,N_r}$$

Two phases and 50 rotor teeth give 1.8°, i.e. 200 steps per revolution.

RESOLUTION AND SPEED

$$S = \frac{360^\circ}{\beta}\ \text{steps/rev}, \qquad n = \frac{\beta f}{6}\ \text{rpm}$$

f = stepping rate in steps (pulses) per second, β in degrees. Half-step drive halves β and doubles S.

CASCADE LOOP, INNER LOOP REDUCED FIRST

$$G_2^{cl} = \frac{G_{c2}G_vG_{p2}}{1 + G_{c2}G_vG_{p2}H_2}, \qquad \frac{Y_1}{R_1} = \frac{G_{c1}G_2^{cl}G_{p1}}{1 + G_{c1}G_2^{cl}G_{p1}H_1}$$

Subscript 1 = outer (primary), 2 = inner (secondary). G_v = valve, H = measuring elements. Convention: negative feedback in both loops, $Y_2 = G_{p2}G_vU$ and $Y_1 = G_{p1}Y_2$ (the inner variable drives the outer process).

CASCADE CHARACTERISTIC EQUATION

$$1 + G_{c2}G_vG_{p2}H_2 + G_{c1}G_{c2}G_vG_{p2}G_{p1}H_1 = 0$$

Obtained by clearing the fractions above. Apply Routh to this equation for cascade stability limits.

IDEAL FEEDFORWARD CONTROLLER

$$G_{ff}(s) = -\frac{G_d(s)}{G_t(s)\,G_v(s)\,G_p(s)}$$

Obtained by setting the disturbance's total effect $(G_d + G_tG_{ff}G_vG_p)\,D$ to zero. The minus sign makes the correction oppose the disturbance.

FEEDFORWARD FOR FOPDT MODELS

$$G_{ff}(s) = -\frac{K_d}{K_p}\,\frac{\tau_p s + 1}{\tau_d s + 1}\,e^{-(\theta_d - \theta_p)s}, \qquad \text{static: } K_{ff} = -\frac{K_d}{K_p}$$

For $G_p = K_p e^{-\theta_p s}/(\tau_p s + 1)$ and $G_d = K_d e^{-\theta_d s}/(\tau_d s + 1)$ with $G_t = G_v = 1$, the result is a lead–lag plus delay. It is exactly realisable only if $\theta_d \geq \theta_p$. If $\theta_d < \theta_p$ it would need a prediction e^{+sT}, so the delay term is dropped. A right-half-plane zero in G_p would make the ideal controller unstable, and an approximation is used instead.

RATIO STATION SETTING

$$F_{c,\text{sp}} = R\,F_w, \qquad K_R = R\,\frac{F_{w,\max}}{F_{c,\max}}$$

K_R is the setting of a ratio station working on normalised (0–1) signals from linear transmitters with spans 0 to $F_{w,\max}$ and 0 to $F_{c,\max}$. With unlinearised orifice (square-law) signals, the setting becomes K_R^2.

SCAN TIMING

$$t_{\text{worst}} \approx 2\,t_{\text{scan}}\ (+\ \text{I/O delays}), \qquad t_{\text{pulse}} > t_{\text{scan}}\ \text{for guaranteed detection}$$

An input that changes just after the input scan is read only on the next scan and written out at the end of that scan, which takes almost two scans. A pulse shorter than one scan can fall between two input scans and be missed entirely. Input-filter and output-module delays add to both figures.

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