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GATE DA

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Calculus and Optimization Formula Sheet

INSIDE

- 44 formulas in 3 topics
- When to use each one
- Taken from the free topic notes on iaeacademy.in

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Limits, continuity and differentiability

LIMIT EXISTS IFF BOTH ONE-SIDED LIMITS AGREE

$$\lim_{x\to a} f(x) = L \iff \lim_{x\to a^-} f(x) = \lim_{x\to a^+} f(x) = L$$

Use one-sided limits for $|x|$, floor, piecewise functions and any function that changes formula at a.

STANDARD TRIGONOMETRIC LIMITS

$$\lim_{x\to 0}\frac{\sin x}{x} = 1, \qquad \lim_{x\to 0}\frac{\tan x}{x} = 1, \qquad \lim_{x\to 0}\frac{1-\cos x}{x^2} = \frac{1}{2}$$

Only for x in radians and $x \to 0$. For $x \to \infty$, $\sin x/x \to 0$ instead.

STANDARD EXPONENTIAL AND LOGARITHMIC LIMITS

$$\lim_{x\to 0}\frac{e^x-1}{x} = 1, \qquad \lim_{x\to 0}\frac{a^x-1}{x} = \ln a \ (a>0), \qquad \lim_{x\to 0}\frac{\ln(1+x)}{x} = 1$$

Each is a $0/0$ form that equals a derivative at 0 (of e^x, a^x and $\ln(1+x)$ respectively).

LIMITS DEFINING E AND THE BINOMIAL LIMIT

$$\lim_{x\to\pm\infty}\left(1+\frac{1}{x}\right)^x = e, \qquad \lim_{x\to 0}(1+x)^{1/x} = e, \qquad \lim_{x\to 0}\frac{(1+x)^n-1}{x} = n$$

Here n is any real number. More generally $\lim_{x\to\infty}(1+k/x)^x = e^k$ for any real k.

THE $1^\INFTY$ FORM

$$\lim_{x\to a} f(x)^{g(x)} = \exp\Big(\lim_{x\to a} g(x)\,[f(x)-1]\Big)$$

Applies when $f(x) > 0$ near a, $f(x) \to 1$, $g(x) \to \pm\infty$, and the limit inside the exponential exists.

EXPANSIONS NEAR $X=0$: EXPONENTIAL AND LOGARITHM

$$e^x = 1 + x + \frac{x^2}{2!} + \frac{x^3}{3!} + \cdots, \qquad \ln(1+x) = x - \frac{x^2}{2} + \frac{x^3}{3} - \cdots$$

The e^x series holds for all real x; the $\ln(1+x)$ series holds for $-1 < x \le 1$.

EXPANSIONS NEAR $X=0$: TRIGONOMETRIC

$$\sin x = x - \frac{x^3}{3!} + \cdots, \qquad \cos x = 1 - \frac{x^2}{2!} + \frac{x^4}{4!} - \cdots, \qquad \tan x = x + \frac{x^3}{3} + \cdots$$

The $\sin x$ and $\cos x$ series hold for all real x; the $\tan x$ series for $|x| < \pi/2$. In a $0/0$ limit at 0, replace each function by enough terms and compare the lowest powers: for example $\dfrac{x-\sin x}{x^3} = \dfrac{x^3/6 - \cdots}{x^3} \to \dfrac{1}{6}$.

L'HOPITAL'S RULE

$$\lim_{x\to a}\frac{f(x)}{g(x)} = \lim_{x\to a}\frac{f'(x)}{g'(x)}$$

Needs: f/g of the form $0/0$ or $\pm\infty/\pm\infty$ as $x \to a$; f, g differentiable with $g'(x) \neq 0$ on a punctured neighbourhood of a; and the limit of f'/g' (the limit on the right of the equation) to exist, finite or infinite. It also holds for one-sided limits and for $a = \pm\infty$. Differentiate numerator and denominator separately; do not use the quotient rule.

GROWTH ORDER

$$\lim_{x\to\infty}\frac{(\ln x)^m}{x^p} = 0, \qquad \lim_{x\to\infty}\frac{x^p}{e^x} = 0 \qquad (p>0,\ m>0)$$

Exponentials beat powers and powers beat logarithms as $x \to \infty$.

CONTINUITY AT A POINT

$$f \text{ is continuous at } a \iff \lim_{x \to a} f(x) = f(a)$$

Three things must hold: $f(a)$ is defined, the limit exists and is finite, and the two are equal. Equivalently, left limit = right limit = $f(a)$.

INTERMEDIATE VALUE THEOREM

$$f \text{ continuous on } [a,b],\ f(a) \neq f(b),\ y \text{ strictly between } f(a) \text{ and } f(b) \ \Rightarrow\ \exists\, c \in (a,b) :\ f(c) = y$$

Used to show a root exists, for example $f(a)f(b) < 0$ gives a root of f in (a,b).

EXTREME VALUE THEOREM

$$f \text{ continuous on the closed bounded interval } [a,b] \ \Rightarrow\ f \text{ attains a global maximum and a global minimum on } [a,b]$$

Both conditions matter: on the open interval $(0,1)$, $f(x) = 1/x$ is continuous but has no maximum.

DERIVATIVE FROM FIRST PRINCIPLES

$$f'(a) = \lim_{h \to 0} \frac{f(a+h) - f(a)}{h} = \lim_{x \to a} \frac{f(x) - f(a)}{x - a}$$

f is differentiable at a when this limit exists and is finite.

LEFT AND RIGHT DERIVATIVES

$$f'_-(a) = \lim_{h \to 0^-} \frac{f(a+h) - f(a)}{h}, \qquad f'_+(a) = \lim_{h \to 0^+} \frac{f(a+h) - f(a)}{h}$$

f is differentiable at a if and only if both exist, are finite and $f'_-(a) = f'_+(a)$.

DIFFERENTIABILITY IMPLIES CONTINUITY

$$f \text{ differentiable at } a \ \Rightarrow\ f \text{ continuous at } a$$

The converse is false. Hence a function that is not continuous at a cannot be differentiable at a.

PRODUCT AND QUOTIENT RULES

$$(uv)' = u'v + uv', \qquad \left(\frac{u}{v}\right)' = \frac{u'v - uv'}{v^2}\ (v \neq 0)$$

For u, v differentiable at the point.

CHAIN RULE

$$\frac{d}{dx} f\big(g(x)\big) = f'\big(g(x)\big)\, g'(x)$$

For g differentiable at x and f differentiable at $g(x)$.

LOGARITHMIC DIFFERENTIATION

$$\frac{d}{dx}\, u(x)^{v(x)} = u^v \left(v' \ln u + \frac{v\, u'}{u} \right)$$

For $u > 0$ and u, v differentiable. For example $\frac{d}{dx} x^x = x^x(1 + \ln x)$ for $x > 0$.

ROLLE'S THEOREM

$$f \text{ continuous on } [a,b],\ \text{differentiable on } (a,b),\ f(a) = f(b) \ \Rightarrow\ \exists\, c \in (a,b) :\ f'(c) = 0$$

All three conditions are needed. Typical use: if f is continuous on the closed interval between two roots and differentiable inside it, then f' has a root strictly between them.

LAGRANGE'S MEAN VALUE THEOREM

$$f \text{ continuous on } [a,b],\ \text{differentiable on } (a,b) \ \Rightarrow\ \exists\, c \in (a,b) :\ f'(c) = \frac{f(b) - f(a)}{b - a}$$

Some point has tangent slope equal to the average slope over $[a,b]$. Rolle's theorem is the special case $f(a) = f(b)$.

CAUCHY'S MEAN VALUE THEOREM

$$f, g \text{ continuous on } [a,b],\ \text{differentiable on } (a,b),\ g'(x) \neq 0 \text{ on } (a,b) \ \Rightarrow\ \exists\, c \in (a,b):\ \frac{f'(c)}{g'(c)} = \frac{f(b)-f(a)}{g(b)-g(a)}$$

The condition $g' \neq 0$ ensures $g(b) \neq g(a)$ (by Rolle's theorem). Taking $g(x) = x$ gives Lagrange's theorem.

Taylor series

TAYLOR SERIES ABOUT X = A

$$f(x) = \sum_{n=0}^{\infty} \frac{f^{(n)}(a)}{n!}(x-a)^n = f(a) + f'(a)(x-a) + \frac{f''(a)}{2!}(x-a)^2 + \cdots$$

Valid for those x where the series converges to $f(x)$; convergence is not automatic, so state the range of x.

MACLAURIN SERIES (A = 0)

$$f(x) = \sum_{n=0}^{\infty} \frac{f^{(n)}(0)}{n!}x^n$$

The coefficient of x^n is $f^{(n)}(0)/n!$, so $f^{(n)}(0) = n! \times (\text{coefficient of } x^n)$.

EXPONENTIAL

$$e^x = \sum_{n=0}^{\infty} \frac{x^n}{n!} = 1 + x + \frac{x^2}{2!} + \frac{x^3}{3!} + \cdots$$

Converges for all real x.

SINE

$$\sin x = \sum_{n=0}^{\infty} \frac{(-1)^n x^{2n+1}}{(2n+1)!} = x - \frac{x^3}{3!} + \frac{x^5}{5!} - \cdots$$

Converges for all real x; only odd powers appear.

COSINE

$$\cos x = \sum_{n=0}^{\infty} \frac{(-1)^n x^{2n}}{(2n)!} = 1 - \frac{x^2}{2!} + \frac{x^4}{4!} - \cdots$$

Converges for all real x; only even powers appear.

NATURAL LOGARITHM

$$\ln(1+x) = \sum_{n=1}^{\infty} \frac{(-1)^{n+1} x^n}{n} = x - \frac{x^2}{2} + \frac{x^3}{3} - \cdots$$

Valid for $-1 < x \leq 1$. At $x = 1$ it gives $\ln 2$; at $x = -1$ it diverges.

BINOMIAL SERIES

$$(1+x)^m = \sum_{k=0}^{\infty} \binom{m}{k} x^k = 1 + mx + \frac{m(m-1)}{2!}x^2 + \cdots, \quad \binom{m}{k} = \frac{m(m-1)\cdots(m-k+1)}{k!}$$

For any real m it is valid for $|x| < 1$ and diverges for $|x| > 1$ (radius 1) unless m is a non-negative integer, in which case the series stops and is exact for all x. At $x = \pm 1$ convergence depends on m (for example, for $m > 0$ it also converges at both endpoints).

GEOMETRIC SERIES

$$\frac{1}{1-x} = \sum_{n=0}^{\infty} x^n = 1 + x + x^2 + \cdots$$

Valid for $|x| < 1$. Replace x by $-x$ for $1/(1+x) = 1 - x + x^2 - \cdots$.

INVERSE TANGENT

$$\tan^{-1} x = \sum_{n=0}^{\infty} \frac{(-1)^n x^{2n+1}}{2n+1} = x - \frac{x^3}{3} + \frac{x^5}{5} - \cdots$$

Valid for $-1 \le x \le 1$; obtained by integrating $1/(1+t^2) = 1 - t^2 + t^4 - \cdots$ from 0 to x. At $x = 1$ it gives $\pi/4$.

TANGENT (FIRST TERMS)

$$\tan x = x + \frac{x^3}{3} + \frac{2x^5}{15} + \cdots$$

Valid for $|x| < \pi/2$; only odd powers appear. The first two terms are usually enough for limits as $x \to 0$.

RADIUS OF CONVERGENCE (RATIO TEST)

$$R = \lim_{n\to\infty} \left| \frac{c_n}{c_{n+1}} \right| \quad \text{for} \quad \sum_{n=0}^{\infty} c_n (x-a)^n$$

When this limit exists (it may be ∞), the series converges absolutely for $|x-a| < R$ and diverges for $|x-a| > R$; the endpoints $|x-a| = R$ must be checked separately. For a series with only odd or only even powers, apply the ratio test to consecutive non-zero terms instead.

TAYLOR'S THEOREM WITH LAGRANGE REMAINDER

$$f(x) = P_n(x) + R_n(x), \qquad R_n(x) = \frac{f^{(n+1)}(c)}{(n+1)!}(x-a)^{n+1} \text{ for some } c \text{ strictly between } a \text{ and } x$$

Needs $f^{(n)}$ continuous on the closed interval between a and x and $f^{(n+1)}$ to exist on the open interval (having $n+1$ continuous derivatives is enough). Hence $|R_n(x)| \le \frac{M}{(n+1)!}|x-a|^{n+1}$, where M is the largest value of $|f^{(n+1)}|$ on that interval.

ALTERNATING SERIES ERROR BOUND

$$|\text{error after } n \text{ terms}| \le |\text{first omitted term}|$$

Applies when, from the first omitted term on, the terms alternate in sign and decrease in magnitude to 0; the error then has the same sign as the first omitted term. Examples: $\ln(1+x)$ for $0 < x \le 1$; $\sin x$, $\cos x$ and $\tan^{-1} x$ for $|x| \le 1$.

HIGHER-ORDER DERIVATIVE TEST

$$f'(a) = f''(a) = \cdots = f^{(n-1)}(a) = 0, \qquad f^{(n)}(a) \neq 0 \quad (n \ge 2)$$

Assume $f^{(n)}$ is continuous near a. If n is even: local minimum when $f^{(n)}(a) > 0$, local maximum when $f^{(n)}(a) < 0$. If n is odd: no local extremum at a. For $n = 2$ this is the second-derivative test.

Maxima, minima and single-variable optimisation

CRITICAL POINT

$$f'(c) = 0 \quad \text{or} \quad f'(c) \text{ does not exist}, \ c \text{ interior to the domain}$$

Necessary condition: if f has a local extremum at an interior point c and $f'(c)$ exists, then $f'(c) = 0$. So every interior local extremum is a critical point. The converse is false: $f(x) = x^3$ has $f'(0) = 0$ but no extremum at 0.

MONOTONICITY FROM THE SIGN OF THE DERIVATIVE

$$f'(x) > 0 \text{ for all } x \in I \Rightarrow f \text{ strictly increasing on } I, \qquad f'(x) < 0 \text{ for all } x \in I \Rightarrow f \text{ strictly decreasing on } I$$

I is an interval, f is continuous on I and differentiable at its interior points. $f' \geq 0$ on I gives non-decreasing; if in addition $f' = 0$ only at isolated points, f is still strictly increasing. The converse of the first implication fails: x^3 is strictly increasing on $\mathbb{R}$ although $f'(0) = 0$.

SECOND-DERIVATIVE TEST

$$f'(c) = 0: \quad f''(c) > 0 \Rightarrow \text{local min}, \qquad f''(c) < 0 \Rightarrow \text{local max}$$

Needs $f'(c) = 0$ and $f''(c)$ to exist, so it cannot be used at a critical point where $f'(c)$ does not exist. If $f''(c) = 0$ the test is inconclusive and you must use the first-derivative test or higher derivatives.

EXTREME VALUE THEOREM

$$f \text{ continuous on } [a,b] \Rightarrow f \text{ attains a global maximum and a global minimum on } [a,b]$$

Both conditions matter: continuity, and a closed bounded interval. On an open interval an extremum may not exist, for example $f(x) = x$ on $(0,1)$ has neither. A discontinuous function on a closed interval may also fail, for example $f(x) = x$ for $0 \leq x < 1$ with $f(1) = 0$ has no maximum on $[0,1]$.

CONVEX AND CONCAVE FUNCTIONS (TWICE DIFFERENTIABLE ON AN INTERVAL I)

$$f''(x) \geq 0 \text{ for all } x \in I \iff f \text{ convex on } I, \qquad f''(x) \leq 0 \text{ for all } x \in I \iff f \text{ concave on } I$$

Convex (also called concave up) means $f(\lambda x + (1-\lambda)y) \leq \lambda f(x) + (1-\lambda) f(y)$ for all $x, y \in I$ and $\lambda \in [0,1]$. For differentiable f this is equivalent to the graph lying on or above every tangent line: $f(y) \geq f(x) + f'(x)(y - x)$ for all $x, y \in I$. $f'' > 0$ on I makes f strictly convex, but not conversely: x^4 is strictly convex although $f''(0) = 0$.

POINT OF INFLECTION

$$f \text{ continuous at } c \text{ and } f'' \text{ changes sign at } c \;\Rightarrow\; c \text{ is a point of inflection}$$

A point of inflection is a point c where f is continuous and the graph changes from convex to concave or the reverse. If $f''(c)$ exists at an inflection point then $f''(c) = 0$. The converse is false: $f(x) = x^4$ has $f''(0) = 0$ but $f'' \geq 0$ on both sides, so 0 is not an inflection point. An inflection point can also occur where f'' does not exist: $f(x) = x|x|$ has $f''(x) = -2$ for $x < 0$ and 2 for $x > 0$, with $f''(0)$ undefined.

SINGLE CRITICAL POINT ON AN INTERVAL

$$f \text{ continuous on an interval } I,\ c \text{ the only critical point in } I,\ c \text{ a local max} \Rightarrow c \text{ is the global max on } I$$

The same holds with min in place of max. Reason: if some $x_1 \in I$ had $f(x_1) > f(c)$, f would have a local minimum strictly between c and x_1, which would be a second critical point.

GRADIENT DESCENT UPDATE (ONE VARIABLE)

$$x_{k+1} = x_k - \eta\, f'(x_k)$$

The iterate stays put exactly when $f'(x_k) = 0$, that is at a critical point. That point need not be a minimum if the iterate lands on it exactly. On a non-convex f the iteration can settle at a local minimum that is not global; which one depends on x_0 and η.

GRADIENT DESCENT ON A QUADRATIC

$$f(x) = \tfrac{a}{2}(x - x^*)^2,\ a > 0: \qquad x_{k+1} - x^* = (1 - \eta a)\,(x_k - x^*) \;\Rightarrow\; x_k - x^* = (1 - \eta a)^k (x_0 - x^*)$$

Converges to x^* from every starting point if and only if $|1 - \eta a| < 1$, that is $0 < \eta < 2/a$. With $\eta = 1/a$ it reaches x^* in one step. For $1/a < \eta < 2/a$ the iterates alternate sides of x^* while converging. At $\eta = 2/a$ they bounce between x_0 and $2x^* - x_0$ forever, and for $\eta > 2/a$ they diverge (whenever $x_0 \neq x^*$). Here $a = f''$, the constant curvature.

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